J.D.D.E. 14(1) (2002), 85-137 (corrected version)

THE MOTION OF WEAKLY INTERACTING PULSES IN REACTION-DIFFUSION SYSTEMS

DEDICATED TO PROFESSORS MASAYASU MIMURA AND TAKAAKI NISHIDA ON THEIR SIXTIETH BIRTHDAYS

SHIN-ICHIRO EI *

Abstract.
The interaction of stable pulse solutions on $\boldsymbol{R}^1$ is considered when distances between pulses are sufficiently large. We construct an attractive local invariant manifold giving the dynamics of interacting pulses in a mathematically regorous way. The equations describing the flow on the manifold is also given in an explicit form. By it, we can easily analyze the movement of pulses such as repulsiveness, attractivity and/or the existence of bound states of pulses. Interaction of front solutions are also treated in a similar way.

Key words. interaction of pulses, interaction of fronts, reaction-diffusion systems

AMS subject classifications. 35B25, 35K57

1. Introduction.

Reaction diffusion systems have been widely treated in order to study temporal and/or spatial pattern formation problems for various phenomena. Among them, the research for the systems on one dimensional space has extensively progressed. Through the research, many important and interesting solutions describing patterns have been shown and analyzed. Traveling wave solutions are one of the typical examples among them, which stand for propagating spatially localized patterns. The existence and stability of traveling wave solutions have been shown for many models such as the Allen-Cahn equation [18], the FitzHugh-Nagumo system ([30], [17] and their references), the Gray-Scott model ([3]) and so on.

In this paper, we suppose the existence of a stable traveling wave or pulse solution and consider the interaction between them. This problem is crucially related to the pattern formation problem, specially to the time evolutional process of localized patterns. The typical example for this problem is the interaction of fronts in the Allen-Cahn equation

$$u_t = \epsilon^2 u_{xx} + \frac{1}{2}u(1-u^2),\ t>0,\ -\infty < x < \infty. \tag{1.1}$$

(1.1) has a stable stationary front solution $U(x) = \tanh\dfrac{x}{2\epsilon}$ satisfying $U(\pm\infty) = \pm 1$ and $U(0) = 0$, which represents a localized pattern. Since (1.1) has a translation invariance, functions $U(x-l)$ and $U(-x+l)$ are also stable stationary solutions of (1.1) for an arbitrarily fixed constant $l \in \boldsymbol{R}^1$. Solutions close to $U(x-l)$ and $U(-x+l)$ are called *kink* and *antikink* solutions respectively. Then, the problem which we will concern here is the interaction of these kink and antikink solutions. That is, we consider how the dynamics of solution is if the initial data $u(0,x)$ is sufficiently close to $U(x-x_1(0)) + U(-x+x_2(0)) - 1$ with $x_1(0) << x_2(0)$. The dynamics is well known ([21], [2] and [15], [13]) that if $\epsilon > 0$ is sufficiently small, the solution $u(t,x)$ remains close to $U(x-x_1(t)) + U(-x+x_2(t)) - 1$ and $x_1(t)$, $x_2(t)$ are approximately governed by

$$\begin{cases} \dot{x}_1 &=& 12\epsilon e^{-\frac{1}{\epsilon}h}, \\ \dot{x}_2 &=& -12\epsilon e^{-\frac{1}{\epsilon}h}, \end{cases} \tag{1.2}$$

where $h = x_2(t) - x_1(t)$. (1.2) describes the coarsening process of the localized patterns because (1.2) means the attractivity of kink-antikink front solutions.

Similar properties hold for the interaction of front solutions in competition-diffusion systems, which is stated in detail in Section 3.

* Graduate School of Integrated Science, Yokohama City University Yokohama 236-0027, Japan (ei2s@yokohama-cu.ac.jp)

Another important example may be the dynamics of fronts in the Cahn-Hilliard equation. The equation describing the interaction of fronts shows the similar coarsening process to (1.2), which has been extensively studied ([1] and its references). But, we will not touch on it in this paper because it has a conservation law and is considered in a bounded interval. In fact, results in this paper are not directly applicable to the dynamics of fronts for the Cahn-Hilliard equation.

On the other hand, localized pulse like patterns in the Gray-Scott model exhibit self-replicating dynamics under some conditions and spatial patterns become complicated ([25], [24]). Also in this process, the interaction of localized patterns gives great influences to the splitting behaviors, which will be reported in the forthcoming paper [11], [12].

Another important and interesting example is the interaction of nerve impulses. Plural nerve impulses frequently run along a nerve axon at the same time depending on initial impulses. Then, the repulsiveness of nerve impulses plays an important role to transmit informations ([29]). On the other hand, there is a simplified model equation called the FitzHugh-Nagumo system describing the dynamics of nerve impulses. A traveling pulse solution of the model equation corresponds to the nerve impulse. By analyzing the interaction of the traveling pulses, we can give the theoretical basis to the repulsive behaviors. In fact, the interaction of traveling pulse solutions in the FitzHugh-Nagumo system was shown to be repulsive under some assumptions by using a formal analysis and the specialty of the system ([4]), Other dynamics of interacting traveling pulses in the FitzHugh-Nagumo systems have also been reported in various settings (e.g. [31], [16]) while they are all formally derived results.

Thus, to consider the interaction of localized solutions gives the important informations on the evolutional process of patterns, but almost all of the results so far have been due to formal discussions except the interaciton of kink-antikink front solutions for the Allen-Cahn equation.

The purpose of this paper is to give a general criterion on the dynamics of weakly interacting traveling wave or pulse solutions in mathematically rigorous way.

Let us consider reaction-diffusion systems of the form

$$\boldsymbol{u}_t = D\boldsymbol{u}_{xx} + F(\boldsymbol{u}),\ t > 0,\ -\infty < x < \infty, \tag{1.3}$$

where $\boldsymbol{u} \in \boldsymbol{R}^n$, $D = diag(d_1, d_2, \cdots, d_n)$ $(d_j \geq 0)$ and F is a smooth function from $\boldsymbol{R}^n$ to $\boldsymbol{R}^n$. We suppose for (1.3) that:

H1) There exist linearly stable equilibria P^- and P^+ in the ODE

$$\boldsymbol{u}_t = F(\boldsymbol{u}),\ t > 0. \tag{1.4}$$

H2) (1.3) has a traveling wave solution with velocity θ connecting from P^- to P^+. That is, there exist a constant θ, positive constants α, β and a function $P(z)$ satisfying the equation

$$\begin{gathered} 0 = DP_{zz} - \theta P_z + F(P),\ -\infty < z < \infty, \\ |P(z) - P^+| \leq O(e^{-\alpha z})\ (z \to +\infty), \\ |P(z) - P^-| \leq O(e^{\beta z})\ (z \to -\infty). \end{gathered} \tag{1.5}$$

We note that $\boldsymbol{u}(t,x) = P(x + \theta t)$ is a solution of (1.3).

H3) Let a differential operator L be

$$L\boldsymbol{v} = D\boldsymbol{v}_{zz} - \theta\boldsymbol{v}_z + F'(P(z))\boldsymbol{v},\ -\infty < z < \infty$$

for $\boldsymbol{v}(z) \in H^2(\boldsymbol{R}^1)$. Then, the spectrum $\Sigma(L)$ of L is given by $\Sigma(L) = \Sigma_0 \cup \{0\}$, where 0 is a simple eigenvalue and there exists a positive constant $\rho_0 > 0$ such that $\Sigma_0 \subset \{z \in \boldsymbol{C};\ \Re z < -\rho_0\}$.

H3) means the traveling wave solution $P(z)$ is stable in a linearized sense. We call $P(z)$ satisfying the assumptions H1) $\sim$ H3) for a constant θ *stable traveling wave solution with velocity* θ. Many models of reaction-diffusion systems have stable traveling wave solutions in this sense.

Transforming (1.3) by $z = x + \theta t$, we have

$$\boldsymbol{u}_t = \mathcal{L}(\boldsymbol{u}), \tag{1.6}$$

where $\mathcal{L}(\boldsymbol{u}) = D\boldsymbol{u}_{zz} - \theta\boldsymbol{u}_z + F(\boldsymbol{u})$. We note that the stable traveling wave solution $P(z)$ is a stable stationary solution of (1.6).

Let $P(z)$ be a stable traveling wave solution with velocity θ. Throughout this paper, we call $P(z)$ *(stable) 1-pulse solution* when $P^- = P^+$ and *(stable) 1-front solution* when $P^- \neq P^+$, respectively.

Let us consider the interaction of 1-pulse solutions, for example. In this case, we may assume $P^- = P^+ = \mathbf{0}$ without loss of generality, where $\mathbf{0} = (0, 0, \cdots, 0) \in \boldsymbol{R}^n$. If the initial data $\boldsymbol{u}(0, x)$ of (1.3) is close enough to $P(x - y_1^*) + P(x - y_2^*)$ for sufficiently large $y_2^* - y_1^*$, then it is shown by the main results in Section 2 that the solution $\boldsymbol{u}(t, x)$ remains sufficiently close to the function

$$P(x + \theta t - y_1(t)) + P(x + \theta t - y_2(t))$$

as long as $y_2(t) - y_1(t)$ is large enough. $y_j(t)$ $(j = 1, 2)$ are essentially governed by the ordinary differential equation with initial data $y_j(0) = y_j^*$,

$$\left\{ \begin{array}{lcl} \dot{y_1} & = - & \langle\, \mathcal{L}(P(z) + P(z-h)), \boldsymbol{\phi}^* \,\rangle_{L^2}\,, \\ \dot{y_2} & = - & \langle\, \mathcal{L}(P(z) + P(z+h)), \boldsymbol{\phi}^* \,\rangle_{L^2}\,, \end{array} \right. \tag{1.7}$$

where $h = h(t) = y_2(t) - y_1(t)$ and $\boldsymbol{\phi}^*$ is an eigenfunction corresponding to 0 eigenvalue of the adjoint operator L^* of L normalized by $\langle\, P_z, \boldsymbol{\phi}^* \,\rangle_{L^2} = 1$. $\langle\, \cdot, \cdot \,\rangle_{L^2}$ denotes the inner product in $L^2(\boldsymbol{R}^1)$.

REMARK 1.1. *When $\theta = 0$ and the linearized operator L is self-adjoint, corresponding results to (1.7) were obtained by Schatzman [28]. Sandstede [27] has got almost same results as this paper at the same time independently of the author.*

When $P(z)$ converges $\mathbf{0}$ such that

$$\begin{aligned} P(z) &\to e^{-\alpha z}\boldsymbol{a}^+ \ (z \to +\infty), \\ P(z) &\to e^{\beta z}\boldsymbol{a}^- \ (z \to -\infty) \end{aligned}$$

for positive constants α, β and non-zero constant vectors $\boldsymbol{a}^\pm$ in $\boldsymbol{R}^n$, we call *exponentially monotone convergent.* If $P(z)$ converges $\mathbf{0}$ in an exponetially monotone way, then it is shown in Section 2 that the right hand side of (1.7) are written

$$\left\{ \begin{array}{lcl} \langle\, \mathcal{L}(P(z) + P(z-h)), \boldsymbol{\phi}^* \,\rangle_{L^2} & = & M_1 e^{-\beta h}\left(1 + O(e^{-\gamma h})\right), \\ \langle\, \mathcal{L}(P(z) + P(z+h)), \boldsymbol{\phi}^* \,\rangle_{L^2} & = & M_2 e^{-\alpha h}\left(1 + O(e^{-\gamma h})\right) \end{array} \right. \tag{1.8}$$

as $h \to +\infty$ for some constants M_j and $\gamma > 0$. Combining (1.7) and (1.8), we have

$$\dot{h} \sim M_1 e^{-\beta h} - M_2 e^{-\alpha h}. \tag{1.9}$$

Thus, the values of constants M_j are important to determine whether 1-pulses are repulsive or attractive while it has been a difficult thing so far. In Section 2, main results will be stated and the formulas for the explicit form of M_j are given. Applications to several examples including the FitzHugh-Nagumo system, the Gray-Scott model equations will be in Section 3. We shall show in the section that the constants M_j can be calculated in explicit forms and we can know easily the interaction of 1-pulses.

The case of the interaction of stable 1-front solutions is treated in a similar manner to the case of stable 1-pulse solutions, but in a slightly different setting from pulses. It will be mentioned parallel to 1-pulses.

Proofs will be in Sections 4 and 5. The basic tools for proofs are integral manifold theory. To construct the unstable manifold, we use the analogous manner in [1] and [2], but the manner of the construction developed in this paper is fairly generalized to be applicable to quite a general reaction-diffusion systems as (1.3).

2. Main results.

Suppose the assumptions H1) $\sim$ H3).

2.1. Interaction of 1-pulses. In this subsection, we assume equilibria $P^{\pm} = \mathbf{0}$.

Let $P(z)$ be a stable 1-pulse solution of (1.3) with volicity θ and fix it. Let $\boldsymbol{\phi}^*(z)$ be the eigenfunction of L^* normalized by $\langle\, P_z, \boldsymbol{\phi}^* \,\rangle_{L^2} = 1$ as stated in Section 1. We consider (1.6) instead of (1.3). Arbitrarily fixing the number of considered 1-pulses, say $N+1$, we let

$$P(z; \boldsymbol{h}) = P(z) + P(z - z_1) + \cdots + P(z - z_N),$$

where $\boldsymbol{h} = (h_1, h_2, \cdots, h_N)$ for $h_j > 0$ and

$$z_j = z_j(\boldsymbol{h}) = \sum_{k=1}^{k=j} h_k \ \ (j = 1, \cdots, N).$$

Here, we set $z_0 = 0$ for convenience.

Let $\Xi(l)$ be the translation operator given by $(\Xi(l)\boldsymbol{v})(z) = \boldsymbol{v}(z - l)$ for $\boldsymbol{v} \in L^2(\boldsymbol{R}^1)$. Moreover, define the quantity

$$\delta(\boldsymbol{h}) = \sup_{z \in \boldsymbol{R}^1} |\mathcal{L}(P(z; \boldsymbol{h}))|,$$

the set

$$\mathcal{M}(h^*) = \left\{ \Xi(l)P(z; \boldsymbol{h}) \ ; \ l \in \boldsymbol{R}^1, \min \boldsymbol{h} > h^* \ \right\},$$

and functions

$$H_j(\boldsymbol{h}) = \langle\, \mathcal{L}(P(z + z_j; \boldsymbol{h})), \boldsymbol{\phi}^* \,\rangle_{L^2}$$

for $j = 0, 1, \cdots, N$. Here, $\min \boldsymbol{h}$ means $\min\{h_1, h_2, \cdots, h_N\}$ for $\boldsymbol{h} = (h_1, h_2, \cdots, h_N)$. Then, we have

THEOREM 2.1. *There exist positive constants h^*, C_0 and a neighborhood $U = U(h^*)$ of $\mathcal{M}(h^*)$ in $\{\ H^2(\boldsymbol{R}^1)\}^n$ such that if $\boldsymbol{u}(0, \cdot) \in U$, then there exist functions $l(t) \in \boldsymbol{R}^1$ and $\boldsymbol{h}(t) \in \boldsymbol{R}^N$ such that*

$$\|\boldsymbol{u}(t, \cdot) - \Xi(l(t))P(z; \boldsymbol{h}(t))\|_\infty \le C_0 \delta(\boldsymbol{h}(t)) \tag{2.1}$$

holds as long as $\min \boldsymbol{h}(t) > h^*$, where $\boldsymbol{u}(t, z)$ is a solution of (1.6). Functions $l(t) \in \boldsymbol{R}^1$ and $\boldsymbol{h}(t) \in \boldsymbol{R}^N$ satisfy

$$\dot{\boldsymbol{h}} = \boldsymbol{H}(\boldsymbol{h}) + O(\delta^2), \tag{2.2}$$

$$\dot{l} = -H_0(\boldsymbol{h}) + O(\delta^2), \tag{2.3}$$

where $\delta = \delta(\boldsymbol{h}(t))$ and $\boldsymbol{H} = (H_0 - H_1, H_1 - H_2, \cdots, H_{N-1} - H_N)$.

REMARK 2.1. *Since $\mathcal{L}(P(z - z_j)) = \mathbf{0}$ and $\mathcal{L}(\mathbf{0}) = \mathbf{0}$ hold for any z_j, we have $\delta(\boldsymbol{h}) \to 0$ as $\min \boldsymbol{h} \to \infty$. On the other hand, the magnitude of $\max_j H_j(\boldsymbol{h})$ is $O(\delta(\boldsymbol{h}))$, which means $\boldsymbol{H}(\boldsymbol{h})$ in (2.2) or (2.3) are necessarily dominant as long as $\min \boldsymbol{h}$ is large enough.*

REMARK 2.2. *Theorem 2.1 gives the motion of positions $y_j = l + z_j$ $(j = 0, 1, \cdots, N)$ of the j-th pulse by*

$$\begin{aligned} \dot{y_j} &= \dot{l} + \dot{h_1} + \cdots + \dot{h_j} \\ &= -H_j(\boldsymbol{h}) + O(\delta^2). \end{aligned}$$

Consider the ordinary differential system consisting of the principal parts of (2.2)

$$\dot{\boldsymbol{h}} = \boldsymbol{H}(\boldsymbol{h}). \tag{2.4}$$

THEOREM 2.2. *Suppose all of the elements d_j of D are positive. Then, there exist positive constants C_0, C_1 and h^* such that if (2.4) has an equilblium $\overline{\boldsymbol{h}}$ satisfying $\min \overline{\boldsymbol{h}} > h^*$ and the set of* *eigenvalues* $\Sigma(\boldsymbol{H}'(\overline{\boldsymbol{h}})) \subset \{z \in \boldsymbol{C}\ ;\ \Re z < -C_0\delta(\overline{\boldsymbol{h}})\}$*, there exists a stable traveling wave pulse solution* $\overline{P}(z+\overline{\theta}t)$ *of (1.6) such that*

$$\|\overline{P}(z) - P(z;\overline{\boldsymbol{h}})\|_\infty \le C_1\delta(\overline{\boldsymbol{h}})$$

and $\overline{\theta} = H_0(\overline{\boldsymbol{h}}) + O(\delta^2(\overline{\boldsymbol{h}}))$*. Here,* $\boldsymbol{H}'(\overline{\boldsymbol{h}})$ *denotes the linearized matrix of* $\boldsymbol{H}$ *with respect to* $\overline{\boldsymbol{h}}$*.*

If (2.4) has an equilblium $\underline{\boldsymbol{h}}$ *such that* $\min \underline{\boldsymbol{h}} > h^*$ *and the set of eigenvalues* $\Sigma(\boldsymbol{H}'(\underline{\boldsymbol{h}})) \subset \{z \in \boldsymbol{C}\ ;\ \Re z < -C_0\delta(\underline{\boldsymbol{h}})\} \cup \{z \in \boldsymbol{C}\ ;\ \Re z > C_0\delta(\underline{\boldsymbol{h}})\}$ *and at least one eigenvalue of* $\boldsymbol{H}'(\underline{\boldsymbol{h}})$ *is in* $\{z \in \boldsymbol{C}\ ;\ \Re z > C_0\delta(\underline{\boldsymbol{h}})\}$*, there exists an unstable traveling wave pulse solution* $\underline{P}(z+\underline{\theta}t)$ *of (1.6) such that*

$$\|\underline{P}(z) - P(z;\underline{\boldsymbol{h}})\|_\infty \le C_1\delta(\underline{\boldsymbol{h}})$$

and $\underline{\theta} = H_0(\underline{\boldsymbol{h}}) + O(\delta^2(\underline{\boldsymbol{h}}))$*.*

REMARK 2.3. *The results stated in Theorem 2.2 have been already obtained by Sandstede [26]. Theorem 2.2 gives an another proof of the results by using the invariant manifold theory as mentioned in the proof.*

REMARK 2.4. *The situation of Theorem 2.2 occurs e.g. in the case when the tail of 1-pulse* $P(z)$ *is oscillatory. This will be stated in Section 3.*

Now, we can know the explicit forms of functions $H_j(\boldsymbol{h})$ when $P(z)$ converges $\boldsymbol{0}$ in an exponentially monotone way. We note here that we may generically assume the corresponding adjoint eigenfunction $\boldsymbol{\phi}^*$ also converges $\boldsymbol{0}$ in similar decaying rates. That is,

THEOREM 2.3. *Suppose* $P(z)$ *converges* $\boldsymbol{0}$ *satisfying*

$$P(z) = e^{-\alpha z}(\boldsymbol{a}^+ + O(e^{-\gamma z}))\ (z \to +\infty), \tag{2.5}$$

$$P(z) = e^{\beta z}(\boldsymbol{a}^- + O(e^{\gamma z}))\ (z \to -\infty) \tag{2.6}$$

for positive constants α*,* β *and* γ *and non-zero constant vectors* $\boldsymbol{a}^\pm \in \boldsymbol{R}^n$*, and suppose* $\boldsymbol{\phi}^*$ *also converges* $\boldsymbol{0}$ *in an exponentially monotone way such that*

$$\boldsymbol{\phi}^*(z) = e^{-\beta z}(\boldsymbol{b}^+ + O(e^{-\gamma z}))\ (z \to +\infty), \tag{2.7}$$

$$\boldsymbol{\phi}^*(z) = e^{\alpha z}(\boldsymbol{b}^- + O(e^{\gamma z}))\ (z \to -\infty) \tag{2.8}$$

for non-zero constant vectors $\boldsymbol{b}^\pm \in \boldsymbol{R}^n$*. Then, functions* $H_j(\boldsymbol{h})$ *are represented by*

$$H_j(\boldsymbol{h}) = \left(M_\beta e^{-\beta h_{j+1}} + M_\alpha e^{-\alpha h_j}\right)\left(1 + O(e^{-\gamma' \min \boldsymbol{h}})\right) \quad (j = 1, 2, \cdots, N-1), \tag{2.9}$$

$$H_0(\boldsymbol{h}) = M_\beta e^{-\beta h_1}\left(1 + O(e^{-\gamma' \min \boldsymbol{h}})\right), \tag{2.10}$$

$$H_N(\boldsymbol{h}) = M_\alpha e^{-\alpha h_N}\left(1 + O(e^{-\gamma' \min \boldsymbol{h}})\right), \tag{2.11}$$

for a constant $\gamma' > 0$ *and the constants* M_α*,* M_β *are given by*

$$M_\beta = 2\beta \langle\, \boldsymbol{a}^-, D\boldsymbol{b}^+ \,\rangle - \theta \langle\, \boldsymbol{a}^-, \boldsymbol{b}^+ \,\rangle, \tag{2.12}$$

$$M_\alpha = 2\alpha \langle\, \boldsymbol{a}^+, D\boldsymbol{b}^- \,\rangle + \theta \langle\, \boldsymbol{a}^+, \boldsymbol{b}^- \,\rangle, \tag{2.13}$$

where $\langle\, \cdot, \cdot \,\rangle$ *stands for the inner product in* $\boldsymbol{R}^n$*.*

COROLLARY 2.1. *Suppose* $P(z)$ *is a symmetric standing pulse, that is,* $P(z)$ *satisfies* $P(z) = P(-z)$ *with* $\theta = 0$*. If* $P(z)$ *converges* $\boldsymbol{0}$ *in an exponentially monotone way, then* $\alpha = \beta$*,* $\boldsymbol{a}^+ = \boldsymbol{a}^-$*, say* $\boldsymbol{a}$*, and* $\boldsymbol{b}^+ = -\boldsymbol{b}^-$*, say* $\boldsymbol{b}$*, hold in (2.5)* $\sim$ *(2.8) of Theorem 2.3. Hence, it follows that*

$$H_j(\boldsymbol{h}) = M_0\left(e^{-\alpha h_{j+1}} - e^{-\alpha h_j}\right)\left(1 + O(e^{-\gamma' \min \boldsymbol{h}})\right) \quad (j = 1, 2, \cdots, N-1), \tag{2.14}$$

$$H_0(\boldsymbol{h}) = M_0 e^{-\alpha h_1}\left(1 + O(e^{-\gamma' \min \boldsymbol{h}})\right), \tag{2.15}$$

$$H_N(\boldsymbol{h}) = -M_0 e^{-\alpha h_N}\left(1 + O(e^{-\gamma' \min \boldsymbol{h}})\right), \tag{2.16}$$

and the constant M_0 is given by

$$M_0 = 2\alpha \langle\, \boldsymbol{a}, D\boldsymbol{b} \,\rangle\,. \tag{2.17}$$

REMARK 2.5. *Theorem 2.1 and Corollary 2.1 show that the movement of interacting symmetric pulses is just determined by the sign of the constant M_0 because the dynamics is given by*

$$\dot{h_j} = M_0\left(2e^{-\alpha h_j} - e^{-\alpha h_{j+1}} - e^{-\alpha h_{j-1}}\right) + o(\delta). \tag{2.18}$$

(2.18) implies $M_0 > 0$ reads the repulsiveness and $M_0 < 0$ does attractivity of pulses.

REMARK 2.6. *In the situation of Corollary 2.1, $\delta = \delta(\boldsymbol{h}) = O(e^{-\alpha \min \boldsymbol{h}})$ holds. Hence, the term $M_0\left(2e^{-\alpha h_j} - e^{-\alpha h_{j+1}} - e^{-\alpha h_{j-1}}\right)$ in (2.18) may not be necessarily dominant when there are more than three pulses and distances between pulses are extremely different. But, we can also prove on the movement of each h_j*

$$\dot{h_j} = M_0\left(2e^{-\alpha h_j} - e^{-\alpha h_{j+1}} - e^{-\alpha h_{j-1}}\right) + o\left(e^{-\alpha h_j} + e^{-\alpha h_{j+1}} + e^{-\alpha h_{j-1}}\right) \tag{2.19}$$

for any distribution of pulses, which means the term $M_0\left(2e^{-\alpha h_j} - e^{-\alpha h_{j+1}} - e^{-\alpha h_{j-1}}\right)$ is necessarily dominant for the movement of the j th pulse. This is proved by slightly modified manner of the proof of Theorem 2.1, but the detail will be mentioned in the forthcoming papars ([12], [14]) because the some parts of the proof are jointly worked in those papers.

2.2. Interaction of standing 1-fronts. For the interaction of 1-fronts, we can consider only the case of the velocity $\theta = 0$. We use the same notations as in the previous subsection with $\theta = 0$, but use x as the space variable instead of z because of $x = z$ in this case.

Let $P(x)$ be a stable 1-front solution of (1.3) with $\theta = 0$ and fix it. Note that $P(-x)$ is also a stable 1-front solution connecting from P^+ to P^-. Assume the number of 1-fronts $N+1 = N^+ + N^-$, where N^+ and N^- are the numbers of 1-fronts of the shapes $P(x)$ and $P(-x)$, respctively. We note that either $N^+ = N^-$ or $N^+ - 1 = N^-$ holds. For $N+1$ 1-fronts, we define

$$P(x;\boldsymbol{h}) = P(x) + P(-(x-x_1)) + P(x-x_2) + \cdots + P\left((-1)^N(x-x_N)\right) - \{N^+P^+ + (N^- - 1)P^-\}$$

if $N^+ = N^-$ and define

$$P(x;\boldsymbol{h}) = P(x) + P(-(x-x_1)) + P(x-x_2) + \cdots + P\left((-1)^N(x-x_N)\right) - \{(N^+ - 1)P^+ + N^-P^-\}$$

if $N^+ - 1 = N^-$, where $\boldsymbol{h} = (h_1, h_2, \cdots, h_N) \in \boldsymbol{R}^N$, $x_j = \sum_{j=1}^{k=j} h_k$ for $j = 1, \cdots, N$ and $x_0 = 0$. Define functions $H_j(\boldsymbol{h})$ $(j = 0, 1, \cdots, N)$ by

$$H_j(\boldsymbol{h}) = \langle\, \mathcal{L}(P(x + x_j; \boldsymbol{h}), \boldsymbol{\phi}^*((-1)^j x) \,\rangle_{L^2}\,.$$

Then, we have

THEOREM 2.4.

Theorems 2.1 and 2.2 hold in the same statements but

$$\dot{h_j} = (-1)^{j+1}(H_{j-1}(\boldsymbol{h}) + H_j(\boldsymbol{h})) + O(\delta^2)\ (j = 1, 2, \cdots, N), \tag{2.20}$$

$$\dot{l} = -H_0(\boldsymbol{h}) + O(\delta^2), \tag{2.21}$$

and

$$\boldsymbol{H} = (H_0 + H_1, -(H_1 + H_2), \cdots, (-1)^{N+1}(H_{N-1} + H_N)).$$

THEOREM 2.5. *Suppose $P(x)$ converges $P^{\pm}$ as*

$$P(x) - P^+ = e^{-\alpha x}(\boldsymbol{a}^+ + O(e^{-\gamma x}))\ (x \to +\infty), \tag{2.22}$$

$$P(x) - P^- = e^{\beta x}(\boldsymbol{a}^- + O(e^{\gamma x}))\ (x \to -\infty) \tag{2.23}$$

for positive constants α, β *and* γ *and non-zero constant vectors* $\boldsymbol{a}^{\pm} \in \boldsymbol{R}^n$, *and suppose* $\boldsymbol{\phi}^*$ *converges* $\boldsymbol{0}$ *in an exponentially monotone way such that*

$$\boldsymbol{\phi}^*(x) = e^{-\alpha x}(\boldsymbol{b}^+ + O(e^{-\gamma x}))\ (x \to +\infty), \tag{2.24}$$

$$\boldsymbol{\phi}^*(x) = e^{\beta x}(\boldsymbol{b}^- + O(e^{\gamma x}))\ (x \to -\infty) \tag{2.25}$$

for non-zero constant vectors $\boldsymbol{b}^{\pm} \in \boldsymbol{R}^n$. *Then, functions* $H_j(\boldsymbol{h})$ *are represented by*

$$H_{2j-1}(\boldsymbol{h}) = \left(M^+ e^{-\alpha h_{2j-1}} - M^- e^{-\beta h_{2j}}\right)\left(1 + O(e^{-\gamma' \min \boldsymbol{h}})\right) \quad (j = 1, 2, \cdots, N^+), \tag{2.26}$$

$$H_{2j}(\boldsymbol{h}) = \left(M^+ e^{-\alpha h_{2j+1}} - M^- e^{-\beta h_{2j}}\right)\left(1 + O(e^{-\gamma' \min \boldsymbol{h}})\right) \quad (j = 1, 2, \cdots, N^-), \tag{2.27}$$

$$H_0(\boldsymbol{h}) = M^+ e^{-\alpha h_1}\left(1 + O(e^{-\gamma' \min \boldsymbol{h}})\right), \tag{2.28}$$

$$H_N(\boldsymbol{h}) = \begin{cases} M^+ e^{-\alpha h_N}\left(1 + O(e^{-\gamma' \min \boldsymbol{h}})\right) & (if\ N^+ = N^-), \\ M^- e^{-\beta h_N}\left(1 + O(e^{-\gamma' \min \boldsymbol{h}})\right) & (if\ N^+ - 1 = N^-) \end{cases} \tag{2.29}$$

for a constant $\gamma' > 0$ *and the constants* $M^{\pm}$ *are given by*

$$M^+ = 2\alpha \langle\, \boldsymbol{a}^+, D\boldsymbol{b}^+ \,\rangle, \tag{2.30}$$

$$M^- = 2\beta \langle\, \boldsymbol{a}^-, D\boldsymbol{b}^- \,\rangle. \tag{2.31}$$

3. Applications.

In this section, the notation $f(h) \sim g(h)$ stands for $f(h) = g(h)(1 + o(1))$ as $h \to \infty$.

3.1. Interaction of 1-fronts in the Allen-Cahn equation. In this subsection, we consider the interaction of 1-fronts of

$$u_t = \varepsilon^2 u_{xx} + f(u),\ t > 0,\ -\infty < x < \infty, \tag{3.1}$$

where $u \in \boldsymbol{R}^1$, $f(u) = \frac{1}{2}u(1 - u^2)$ and ε is a sufficiently small positive parameter. Then, (3.1) has a stable standing 1-front $P(x)$ given by

$$P(x) = \tanh \frac{x}{2\varepsilon}$$

as in Section 1. This 1-front is linearly stable and connecting $P^{\pm} = \pm 1$ ([18]).

Let us consider the interaction of two 1-fronts $P(x - l)$ and $P(-(x - l - h))$ for $l,\ h > 0$ by using Theorems 2.4 and 2.5. Let $P(x; h) = P(x) + P(-(x - h)) - 1$. Then, $\delta(h) = \sup_{x \in \boldsymbol{R}^1} |\mathcal{L}(P(x; h))|$ is estimated as $O(e^{-\frac{h}{\epsilon}})$ because the asymptotic form of $P(x)$ as $x \to \infty$ is

$$P(x) \to -2e^{-\frac{x}{\varepsilon}} + 1. \tag{3.2}$$

On the other hand, the eigenfunction $\phi^*(x)$ is easily obtained as

$$\phi^*(x) = \frac{3\varepsilon}{2} P_x(x)$$

together with its asymptotic form

$$\phi^*(x) \to 3e^{-\frac{x}{\varepsilon}}\ (x \to \infty) \tag{3.3}$$

because the linealized operator with respect to $P(x)$ is self-adjoint. Thus, all of necessary quantities in Theorem 2.5 are given by

$$D = \varepsilon^2, \ \alpha = \frac{1}{\varepsilon}, \ \boldsymbol{a}^+ = -2, \ \boldsymbol{b}^+ = 3$$

from (3.2) and (3.3). Hence, $H_j(h)$ are calculated as

$$\begin{aligned} H_0(h), \ H_1(h) &\sim M^+ e^{-\alpha h} = 2\alpha \langle \boldsymbol{a}^+, D\boldsymbol{b}^+ \rangle e^{-\alpha h} \\ &= 2 \cdot \frac{1}{\varepsilon} \cdot (-2) \cdot \varepsilon^2 \cdot 3 \cdot e^{-\frac{1}{\varepsilon}h} \\ &= -12\varepsilon e^{-\frac{1}{\varepsilon}h} \end{aligned}$$

and we get the equation of l and h as

$$\left\{ \begin{array}{rcl} \dot{l} &=& -H_0(h) + O(\delta^2(h)) \sim 12\varepsilon e^{-\frac{1}{\varepsilon}h}, \\ \dot{h} &=& (H_0(h) + H_1(h)) + O(\delta^2(h)) \sim -24\varepsilon e^{-\frac{1}{\varepsilon}h}. \end{array} \right. \tag{3.4}$$

Especially, the equation of the distance h of two 1-fronts is that otained by [13], [2] and [15].

3.2. Interaction of 1-pulses in the FitzHugh-Nagumo system. Let us consider the equation

$$\left\{ \begin{array}{rcl} u_t &=& \varepsilon^2 u_{xx} + f(u) - v, \\ v_t &=& \varepsilon(u - bv), \end{array} \right. \tag{3.5}$$

where $f(u) = u(1-u)(u-a)$ $(0 < a < \frac{1}{2})$, b is small enough that two graphs $v = f(u)$ and $u = bv$ have no intersection except $\mathbf{0} = (0, 0)$. When $\varepsilon > 0$ is suffiently small, it is known (e.g. [17], [30]) that (3.5) has a stable traveling 1-pulse solution, say $P(z) = (\Phi(z), \Psi(z))$, with velocity $\theta = \varepsilon c$ for a positive constant c and converging $\mathbf{0}$ in an exponentially monotone way. Transforming (3.5) by $z = x + \varepsilon ct$, we have

$$\boldsymbol{u}_t = D\boldsymbol{u}_{zz} - \varepsilon c \boldsymbol{u}_z + F(\boldsymbol{u}), \tag{3.6}$$

where $\boldsymbol{u} = (u, v)$, $D = \begin{pmatrix} \varepsilon^2 & 0 \\ 0 & 0 \end{pmatrix}$ and $F(\boldsymbol{u}) = \begin{pmatrix} f(u) - v \\ \varepsilon(u - bv) \end{pmatrix}$.

Let $c = c_0 + \varepsilon c_1 + O(\varepsilon^2)$. By the construction of $P(z)$ (e.g. [30]), the asymptotic form of $P(z)$ is easily known as

$$P(z) \to \left\{ \begin{array}{l} e^{\frac{K_1}{\varepsilon}z} \boldsymbol{a}^- \ (z \to -\infty), \\ e^{-\alpha z} \boldsymbol{a}^+ \ (z \to +\infty) \end{array} \right. \tag{3.7}$$

for a positive constant K_1, $\boldsymbol{a}^- \in \boldsymbol{R}^2$ and

$$\alpha = \frac{1}{c_0} \left(b - \frac{1}{f'(0)} \right) + O(\varepsilon), \tag{3.8}$$

$$\boldsymbol{a}^+ = -K_2 \begin{pmatrix} 1 \\ f'(0) \end{pmatrix} + O(\varepsilon) \in \boldsymbol{R}^2 \tag{3.9}$$

for $K_2 > 0$. Hence, $\delta(h) = \sup_{z \in \boldsymbol{R}^1} |\mathcal{L}(P(z) + P(z-h))|$ is $O(e^{-\alpha h})$.

On the other hand, the eigenfunction $\boldsymbol{\phi}^*(z)$ of the adjoint operator of the linearized equation of (3.6) extensively studied in [4], [9]. By using the result, we can know the asymptotic form of $\boldsymbol{\phi}^*$ as

$$\boldsymbol{\phi}^*(z) \to \left\{ \begin{array}{l} e^{\alpha z} \boldsymbol{b}^- \ (z \to -\infty), \\ e^{-\frac{K_3}{\varepsilon}z} \boldsymbol{b}^+ \ (z \to +\infty) \end{array} \right.$$

for $K_3 > 0$, $\boldsymbol{b}^+ \in \boldsymbol{R}^2$ and

$$\boldsymbol{b}^- = K_4 \begin{pmatrix} \frac{\varepsilon}{f'(0)} + O(\varepsilon^2) \\ -1 + O(\varepsilon) \end{pmatrix} \in \boldsymbol{R}^2 \tag{3.10}$$

for $K_4 > 0$. Hence, Theorems 2.1 and 2.3 yield the equation of pulse distance h as follows:

$$\dot{h} = -M_\alpha e^{-\alpha h} + M_\beta e^{-\frac{K_1}{\varepsilon} h} + O(\delta^2(h)) \sim -M_\alpha e^{-\alpha h}.$$

Here, we ignored the second term in the above equation due to the smallness of ε, We can also obtain the explicit value of M_α by substituting (3.8), (3.9) and (3.10) into (2.12) as

$$\begin{aligned} M_\alpha &= 2\alpha \langle \boldsymbol{a}^+, D\boldsymbol{b}^- \rangle + \varepsilon c \langle \boldsymbol{a}^+, \boldsymbol{b}^- \rangle \\ &= \varepsilon c_0 K_2 K_4 f'(0) + O(\varepsilon^2) \\ &< 0. \end{aligned}$$

This shows two 1-pulses in (3.5) interact repulsively.

3.3. Interaction of 1-pulses with oscillatory tails. In this subsection, we will consider the interaction of two 1-pulses with oscillatory tails such that

$$P(z) \to \Re\left(e^{\lambda^\pm z}\boldsymbol{a}^\pm\right)$$

as $z \to \pm\infty$, where $\boldsymbol{a}^\pm \in \boldsymbol{C}^n$ and $\lambda^+ = -\alpha + i\nu^+$, $\lambda^- = \beta + i\nu^-$ for positive constants α, β, and we assume either $\nu^\pm$ is not zero. We let $\nu^+ \neq 0$ for simplicity and suppose $\alpha < \beta$. This means the right tail of $P(z)$ converges $\mathbf{0}$ slowly and oscillatorily. This setting has been extensively studied for the pulse solution of the FitzHugh-Nagumo system related to multi-pulse solutions (see e.g. [26] and the references).

Let us consider the interaction of two 1-pulses. Then, the equation describing the distance h between 1-pulses is

$$\dot{h} = H_0 - H_1$$

as in Theorem 2.1. By quite a similar way to the proof of theorem 2.3, we can show

$$\begin{aligned} H_0(h) &= \Re\left(e^{-\lambda^- h} M^- \left(1 + O(e^{-\gamma' h})\right)\right), \\ H_1(h) &= \Re\left(e^{\lambda^+ h} M^+ \left(1 + O(e^{-\gamma' h})\right)\right) \end{aligned}$$

for a constant $\gamma' > 0$, where

$$\begin{aligned} M^+ &= \int_{-\infty}^{\infty} e^{\lambda^+ z} \langle F'(P(z)) - F'(\mathbf{0})\boldsymbol{a}^+, \boldsymbol{\phi}^*(z) \rangle \, dz, \\ M^- &= \int_{-\infty}^{\infty} e^{\lambda^- z} \langle F'(P(z)) - F'(\mathbf{0})\boldsymbol{a}^-, \boldsymbol{\phi}^*(z) \rangle \, dz. \end{aligned}$$

Note that the constants M^+ and M^- are well defined because the integrals giving their constants are given as the Fourier transformation because of the form of $\lambda^\pm$. Let $M^\pm = A^\pm + iB^\pm$. Then, we have

$$\begin{aligned} H_0(h) &\sim e^{-\beta h}(A^- \cos \nu^- h - B^- \sin \nu^- h), \\ H_1(h) &\sim e^{-\alpha h}(A^+ \cos \nu^+ h + B^+ \sin \nu^+ h). \end{aligned}$$

Since $\alpha < \beta$ and $\delta(h) = O(e^{-\alpha h})$ in this case, the equation on h is

$$\dot{h} = H_0 - H_1 + O(\delta^2(h)) \sim -H_1 \sim -e^{-\alpha h}(A^+ \cos \nu^+ h + B^+ \sin \nu^+ h) \tag{3.11}$$

for sufficiently large h. From (3.11), we easily find that stable and unstable equilibria appear alternatively in (3.11) satisfying the assumptions of Theorem 2.2 because $\delta(h) = e^{-\alpha h}$. This gives the existence and its stability of double pulse solutions by Theorem 2.2.

We have considered the interaction only of two 1-pulses in this subsection, but the method developed here is easily extended to arbitrarily many numbers of 1-pulses. Thus, we can easily give an another proof on the existence and stability analysis of multiple-pulse solutions.

3.4. Interaction of 1-pulses in the Gray-Scott model on 1-D. We will show the repulsiveness of pulses in the Gray-Scott model equations on one dimension. The model equation here is

$$\left\{\begin{array}{lcl} u_t & = & u_{xx} - uv^2 + A(1-u), \\ v_t & = & D_v v_{xx} - Bv + uv^2, \end{array}\right. \tag{3.12}$$

where A, B and D_v are positive constants. Doelman et. al. [3] showed the existence of a stable standing 1-pulse solution, say $P(x) = (\Phi(x), \Psi(x))$ under the assumptions

$$A = \epsilon^2 a,\ B = \epsilon^\nu b,\ D_v = \epsilon^2$$

for a small parameter $\epsilon > 0$ and positive constants a. b and $0 < \nu < 1$. We consider in this subsection only the case $\nu = \frac{1}{2}$. The profile $P(x)$ is then given by ([3])

$$\begin{array}{lcl} \Phi(x) & = & \epsilon^{\frac{3}{4}}\{p_0 + o(1)\},\ (x \sim 0), \\ \Phi(x) & = & \Phi_0(x) + o(1),\ x > 0, \\ \Psi(x) & = & \epsilon^{-\frac{1}{4}}\{q_0(\eta) + O(\epsilon^{\frac{1}{2}})\} \end{array} \tag{3.13}$$

as $\epsilon \to 0$, where $p_0 = 3b\sqrt{\frac{b}{a}}$, $q_0(\eta) = \frac{1}{2}\sqrt{\frac{b}{a}}\mathrm{sech}^2\left(\frac{\sqrt{b}}{2}\eta\right)$, $\eta = \epsilon^{-\frac{3}{4}}x$ and Φ_0 is the function satisfying

$$0 = \frac{\partial^2 \Phi_0}{\partial x^2} + \epsilon^2 a(1-\Phi_0),\ \Phi_0(0) = \epsilon^{\frac{3}{4}}p_0,\ \Phi_0(\infty) = 1.$$

Let us obtain the adjoint eigenfunction $\boldsymbol{\phi}^*(x) = (\phi^*(x), \psi^*(x))$ corresponding to the zero eigenvalue of the adjoint operator. Here, we note that the following calculations are all easily justified by singular perturbation techniques though we treat it by the formal asymptotic expansion for simplicity.

The equation which should be satisfied by $\boldsymbol{\phi}^*$ is

$$0 = u_{xx} - (\Psi^2 + \epsilon^2 a)u + \Psi^2 v, \tag{3.14}$$

$$0 = \epsilon^2 v_{xx} - 2\Phi\Psi u + (2\Phi\Psi - \epsilon^{\frac{1}{2}}b)v \tag{3.15}$$

with $u(0) = v(0) = 0$ and $u(\infty) = v(\infty) = 0$ because $\boldsymbol{\phi}^*$ is a bounded odd function. It suffices to consider (3.14) and (3.15) only for $x > 0$. Transforming (3.14) and (3.15) by $u = \epsilon^{\frac{3}{4}}\phi$, $\Phi = \epsilon^{\frac{3}{4}}p$, $v = \epsilon^{-\frac{1}{4}}\psi$, $\Psi = \epsilon^{-\frac{1}{4}}q$ and $x = \epsilon^{\frac{3}{4}}\eta$, we have

$$0 = \phi_{\eta\eta} - (\epsilon q^2 + \epsilon^{\frac{7}{2}}a)\phi + q^2\psi, \tag{3.16}$$

$$0 = \psi_{\eta\eta} - 2\epsilon pq\phi + (2pq - b)\psi. \tag{3.17}$$

Suppose $\phi = \phi_0 + o(1)$ and $\psi = \psi_0 + o(1)$ as $\epsilon \to 0$. Then, ϕ_0 and ψ_0 satisfy

$$0 = \frac{\partial^2 \phi_0}{\partial \eta^2} + q_0^2\psi_0, \tag{3.18}$$

$$0 = \frac{\partial^2 \psi_0}{\partial \eta^2} + (2p_0 q_0 - b)\psi_0 \tag{3.19}$$

with $\phi_0(0) = \psi_0(0) = 0$. Substituting p_0 and $q_0(\eta)$ into (3.19), we have

$$0 = \frac{\partial^2 \psi_0}{\partial \eta^2} + b\left\{3\mathrm{sech}^2\left(\frac{\sqrt{b}}{2}\eta\right) - 1\right\}\psi_0$$

which is easily solved (e.g. [22]) as $\psi_0(\eta) = rM_1(\eta)$ for $r \in \boldsymbol{R}^1$, where

$$M_1(\eta) = \mathrm{sech}^2\left(\frac{\sqrt{b}}{2}\eta\right)\tanh\left(\frac{\sqrt{b}}{2}\eta\right).$$

Hence, ϕ_0 is obtained as

$$\phi_0(\eta) = rM_2(\eta),$$

where

$$M_2(\eta) = \int_0^{\eta} \left\{ -\int_0^{\eta'} q_0^2(\eta")M_1(\eta")d\eta" + K_5 \right\} d\eta'$$

and $K_5 = \displaystyle\int_0^{\infty} q_0^2(\eta)M_1(\eta)d\eta$. Let $\zeta^* = M_2(\infty)$, which is a positive constant.

On the other hand, for $x > 0$, we may assume $\Psi = 0$ and $v = 0$ in (3.14), (3.15) because $\Psi(x)$ and $v(x)$ are $O(e^{-Cx/\epsilon^{\frac{3}{4}}})$. Let $u(x) = u_0(x) + O(e^{-Cx/\epsilon^{\frac{3}{4}}})$ as $\epsilon \to 0$. Then, u_0 satisfies

$$0 = \frac{\partial^2 u_0}{\partial x^2} - \epsilon^2 a u_0$$

with $u_0(0) = \epsilon^{\frac{3}{4}} r\zeta^*$ and $u_0(\infty) = 0$. Hence, it follows

$$u_0(x) = \epsilon^{\frac{3}{4}} r\zeta^* e^{-\epsilon\sqrt{a}x}.$$

As a consequence, we have the adjoint eigenfunction $\boldsymbol{\phi}^*(x)$ as

$$\begin{array}{lcl} \phi^*(x) &=& r\epsilon^{\frac{3}{4}}(M_2(\epsilon^{-\frac{3}{4}}x) + o(1)),\ (x \sim 0), \\ \phi^*(x) &=& r\epsilon^{\frac{3}{4}}(\zeta^* e^{-\epsilon\sqrt{a}x} + o(1)),\ (x > 0), \\ \psi^*(x) &=& r\epsilon^{-\frac{1}{4}}(M_1(\epsilon^{-\frac{3}{4}}x) + o(1)) \end{array} \tag{3.20}$$

as $\epsilon \to 0$ and therefore,

$$\langle\, P_x, \boldsymbol{\phi}^* \,\rangle_{L^2} = 2r\epsilon^{-\frac{1}{2}} \left(\int_0^{\infty} M_1(\eta)q_0'(\eta)d\eta + o(1) \right)$$

holds. This means the normalized condition $\langle\, P_x, \boldsymbol{\phi}^* \,\rangle_{L^2} = 1$ yields $r = -r_0\sqrt{\epsilon} < 0$ for $r_0 > 0$. Since the asymptotic forms $P(x)$ and $\boldsymbol{\phi}^*(x)$ are respectively given by

$$P(x) \to e^{-\epsilon\sqrt{a}x}\boldsymbol{a},\ \boldsymbol{\phi}^*(x) \to e^{-\epsilon\sqrt{a}x}\boldsymbol{b},$$

where $\boldsymbol{a} = (-a^+, 0)$ and $\boldsymbol{b} = (\epsilon^{\frac{3}{4}} r\zeta^*, 0)$ for a positive constant a^+, the constant M_0 in Corollary 2.1 is

$$M_0 = 2\alpha \langle\, D\boldsymbol{a}, \boldsymbol{b} \,\rangle = 2\epsilon\sqrt{a}(-a^+)\epsilon^{\frac{3}{4}} r\zeta^* = 2\epsilon^{\frac{7}{4}}\sqrt{a}a^+ r_0\zeta^* > 0$$

due to the positivity of ζ^* and r_0, where $\alpha = \epsilon\sqrt{a}$ and $D = diag(1, D_v)$. Thus, we find the interaction is repulsive.

3.5. Interaction of 1-fronts in competition-diffusion systems. The model equation which we consider here is the following system

$$\left\{ \begin{array}{lcl} u_t &=& u_{xx} + u(1 - u - av), \\ v_t &=& dv_{xx} + v(c - bu - v), \end{array} \right. \tag{3.21}$$

where a, b, c and d are positive constans. For (3.21), it is known by [20] that if $\dfrac{1}{a} < b$ is satisfied, then the kinetics of (3.21) is bistable, that is, (3.21) has two stable equilibria $P^+ = (0, c)$, $P^- = (1, 0)$ and there exists $c = c_0 \in (\dfrac{1}{a}, b)$ such that (3.21) has a unique stable front $P(x)$ with velocity $\theta = 0$ connecting $P^{\pm}$. Fix $c = c_0$ in (3.21) and let $P(x) = (\Phi(x), \Psi(x))$ be the stable standing 1-front with $P(x) \to P^{\pm}$ as $x \to \pm\infty$.

Let us consider the interaction of two 1-fronts $P(x-l)$ and $P(-(x-l-h))$ for $l,\ h>0$. First, we note that

$$\Phi_x(x)<0,\ \Psi_x(x)>0 \tag{3.22}$$

hold for $-\infty<x<\infty$ ([20]). This means the asymptotic form of $P(x)$ is

$$P(x)\to e^{-\alpha x}\boldsymbol{a}^+ + P^+\ (x\to\infty) \tag{3.23}$$

for $\alpha>0$ and a vector $\boldsymbol{a}^+\in\boldsymbol{R}^2$ written by $\boldsymbol{a}^+=(p,-q)$ for positive constans p, q.

On the other hand, it is also known in [20] that the eigenfunction $\boldsymbol{\phi}^*(x)=(\phi^*(x),\psi^*(x))$ of the adjoint operator of the linearized equation of (3.21) satisfies

$$\phi^*(x)\cdot\psi^*(x)<0. \tag{3.24}$$

Since $\boldsymbol{\phi}^*(x)$ is normarized by the condition $\langle\ P_x,\boldsymbol{\phi}^*\ \rangle_{L^2}=1$, (3.22) and (3.24) imply $\phi^*(x)<0$ and $\psi^*(x)>0$ for $-\infty<x<\infty$. Hence, we can find that the asymptotic form of $\boldsymbol{\phi}^*(x)$ is

$$\boldsymbol{\phi}^*(x)\to e^{-\alpha x}\boldsymbol{b}^+\ (x\to\infty), \tag{3.25}$$

where $\boldsymbol{b}^+=(-r,s)\in\boldsymbol{R}^2$ for positive constants r, s. Thus, Theorems 2.4 and 2.5 yield

$$\begin{aligned}H_0(h),\ H_1(h) &\sim M^+e^{-\alpha h}\\ &=2\alpha\ \langle\ \boldsymbol{a}^+,D\boldsymbol{b}^+\ \rangle\ e^{-\alpha h}\\ &=2\cdot\alpha\cdot\langle\ \begin{pmatrix}p\\-q\end{pmatrix},\begin{pmatrix}1&0\\0&d\end{pmatrix}\begin{pmatrix}-r\\s\end{pmatrix}\ \rangle\ e^{-\alpha h}\\ &=-2\alpha(pr+dqs)e^{-\alpha h}\\ &<0\end{aligned}$$

and the equation of the distance h

$$\dot{h}\sim H_0(h)+H_1(h)\sim 2M^+e^{-\alpha h}=-4\alpha(pr+dqs)e^{-\alpha h}<0.$$

This shows the attractivity of 1-fronts.

4. Proofs of Theorems.

In this section, we will devote ourselves to the proofs of theorems in Section 2.1. Theorems in Section 2.2 are all proved in quite a similar manner to this section and we omit the proofs.

In this and the following sections, C denotes a generic positive constant independent of $\boldsymbol{h}$ with sufficiently large $\min\boldsymbol{h}$ and we take h^* sufficiently large as required in following lemmas.

First, we consider the case that all of the elements of $D=diag(d_1,d_2,\cdots,d_n)$ are positive.

4.1. Preliminaries for the proofs of Theorems 2.1, 2.2.

Let $X=\left\{L^2(\boldsymbol{R}^1)\right\}^n$ with the norm $\|\cdot\|$ and

$$\begin{aligned}L(\boldsymbol{h})\boldsymbol{v}&=D\boldsymbol{v}_{zz}-\theta\boldsymbol{v}_z+F'(P(\cdot;\boldsymbol{h}))\boldsymbol{v},\\ L^*(\boldsymbol{h})\boldsymbol{v}&=D\boldsymbol{v}_{zz}+\theta\boldsymbol{v}_z+{}^tF'(P(\cdot;\boldsymbol{h}))\boldsymbol{v}\end{aligned}$$

for $\boldsymbol{v}\in\left\{H^2(\boldsymbol{R}^1)\right\}^n$, where ${}^tF'$ means the transposed matrix of F'. These operators are sectrial in X.

PROPOSITION 4.1. *There exist positive constants C and h^* such that for $\boldsymbol{h}$ with $\min\boldsymbol{h}>h^*$, the operator $L(\boldsymbol{h})$ has $N+1$ semi-simple eigenvalues $\{\lambda_j(\boldsymbol{h})\}_{j=0,\cdots,N}$ with $|\lambda_j(\boldsymbol{h})|\le C\delta(\boldsymbol{h})$. Multiple eigenvalues are repeated as many times as their multiplicity indicates. Other spectra of $L(\boldsymbol{h})$ are in the left hand side of $z=-\rho_0$ for a positive constant ρ_0.*

Proof. . See e.g. [23] and [26]. ∎

Let $E(\boldsymbol{h})$ be the eigenspace corresponding to eigenvalues $\{\lambda_j(\boldsymbol{h})\}_{j=0,\cdots,N}$. The adjoint operator $L^*(\boldsymbol{h})$ has also similar $N+1$ semi-simple eigenvalues $\{\lambda_j^*(\boldsymbol{h})\}_{j=0,\cdots,N}$ with $|\lambda_j^*(\boldsymbol{h})|\le C\delta(\boldsymbol{h})$. Let $E^*(\boldsymbol{h})$ be the eigenspace corresponding to eigenvalues $\{\lambda_j^*(\boldsymbol{h})\}_{j=0,\cdots,N}$.

PROPOSITION 4.2. *$E(\boldsymbol{h})$ and $E^*(\boldsymbol{h})$ are spaned by $N+1$ functions $\{\boldsymbol{\phi}_j(\boldsymbol{h})(\cdot)\}$ and $\{\boldsymbol{\phi}_j^*(\boldsymbol{h})(\cdot)\}$ ($j = 0,\cdots,N$) respectively such that for $j = 0,\cdots,N$,*

$$\boldsymbol{\phi}_j(\boldsymbol{h})(z) = P_z(z - z_j) + O(\delta), \tag{4.1}$$

$$\boldsymbol{\phi}_j^*(\boldsymbol{h})(z) = \boldsymbol{\phi}^*(z - z_j) + O(\delta), \tag{4.2}$$

$$\langle\ \boldsymbol{\phi}_j(\boldsymbol{h}), \boldsymbol{\phi}_k^*(\boldsymbol{h})\ \rangle_{L^2} = 0\ (j \neq k), \tag{4.3}$$

$$\langle\ \boldsymbol{\phi}_j(\boldsymbol{h}), \boldsymbol{\phi}_j^*(\boldsymbol{h})\ \rangle_{L^2} = 1 \tag{4.4}$$

hold, where $\delta = \delta(\boldsymbol{h})$, $z_j = z_j(\boldsymbol{h})$, and $O(\delta)$ mean in this lemma $\|O(\delta)\|_{H^2} \leq C\delta$.

Proofs of Propositions 4.1 and 4.2 will be given together in Section 5 for convenience while the proof of Proposition 4.1 has been already shown in some papers ([23] and [26]).

Now, we fix $h^* > 0$ large enough such that Propositions 4.1 and 4.2 hold.

Let operators $Q(\boldsymbol{h})$ and $R(\boldsymbol{h})$ be the projection from X to $E(\boldsymbol{h})$ and $R(\boldsymbol{h}) = Id - Q(\boldsymbol{h})$ respectively, where Id is the identity on X. Let $E^\perp(\boldsymbol{h}) = R(\boldsymbol{h})X$. Note that $E^\perp(\boldsymbol{h})$ is characterized by

$$E^\perp(\boldsymbol{h}) = \{\boldsymbol{v} \in X\ ;\ \langle\ \boldsymbol{v}, \boldsymbol{\phi}_j^*(\boldsymbol{h})\ \rangle_{L^2} = 0\ (j = 0,\cdots,N)\}.$$

Let $\boldsymbol{h}^* = (h^*, h^*,\cdots,h^*)$, $\widehat{\boldsymbol{h}} = (\widehat{h}_1, \widehat{h}_2,\cdots,\widehat{h}_N) \in \boldsymbol{R}^N$ with $\min\widehat{\boldsymbol{h}} > h^*$ and $\delta^* = \delta(\boldsymbol{h}^*)$, $\widehat{\delta} = \delta(\widehat{\boldsymbol{h}})$. We define a map $\Pi(\boldsymbol{h})$ from $E^\perp(\widehat{\boldsymbol{h}})$ to $E^\perp(\boldsymbol{h})$ for $\boldsymbol{h} = (h_1, h_2,\cdots,h_N) \in \boldsymbol{R}^N$ with $h_j > \widehat{h}_j$ as follows:

Let $\Theta(r;\boldsymbol{h}) = r\boldsymbol{h} + (1-r)\widehat{\boldsymbol{h}}$ $(0 \leq r \leq 1)$ and let

$$S(r;\boldsymbol{h})\boldsymbol{v} = -\sum_{j,k=0}^{N}(h_j - \widehat{h}_j)\,\langle\ \boldsymbol{v}, \frac{\partial}{\partial h_j}\boldsymbol{\phi}_k^*(\Theta(r))\ \rangle_{L^2}\,\boldsymbol{\phi}_k(\Theta(r))$$

for $\boldsymbol{v} \in X$, where $\Theta(r) = \Theta(r;\boldsymbol{h})$. Define the map $\Pi(\boldsymbol{h})$ by $\Pi(\boldsymbol{h})\boldsymbol{v}_0 = \boldsymbol{v}(1)$, where $\boldsymbol{v}(r)$ is a solution of

$$\left\{\begin{array}{rcl}\dfrac{d\boldsymbol{v}}{dr} &=& S(r;\boldsymbol{h})\boldsymbol{v},\\ \boldsymbol{v}(0) &=& \boldsymbol{v}_0 \in X.\end{array}\right. \tag{4.5}$$

LEMMA 4.1.
The map $\Pi(\boldsymbol{h})$ is a homeomorphism from $E^\perp(\widehat{\boldsymbol{h}})$ to $E^\perp(\boldsymbol{h})$.

Proof. Let $\boldsymbol{v}(r)$ be a solution of (4.5) with $\boldsymbol{v}_0 \in E^\perp(\widehat{\boldsymbol{h}})$. We simply write $Q(r) = Q(\Theta(r;\boldsymbol{h}))$, $R(r) = R(\Theta(r;\boldsymbol{h}))$ and so on. Let $\boldsymbol{v}_1(r) = Q(r)\boldsymbol{v}(r)$ and $\boldsymbol{v}_2(r) = R(r)\boldsymbol{v}(r)$. Then, $\boldsymbol{v}_1(r)$ satisfies

$$\begin{aligned}\frac{d\boldsymbol{v}_1}{dr} &= \frac{d}{dr}\{Q(r)\boldsymbol{v}(r)\}\\ &= \frac{dQ}{dr}(r)\boldsymbol{v}(r) + Q(r)\frac{d\boldsymbol{v}}{dr}(r)\\ &= \frac{dQ}{dr}(r)\boldsymbol{v}(r) + Q(r)S(r)\boldsymbol{v}(r)\\ &= \frac{dQ}{dr}(r)(\boldsymbol{v}_1(r) + \boldsymbol{v}_2(r)) + Q(r)S(r)(\boldsymbol{v}_1(r) + \boldsymbol{v}_2(r))\\ &= \frac{dQ}{dr}(r)(\boldsymbol{v}_1(r) + \boldsymbol{v}_2(r)) + S(r)\boldsymbol{v}_1(r) - \frac{dQ}{dr}(r)\boldsymbol{v}_2(r)\\ &= \frac{dQ}{dr}(r)\boldsymbol{v}_1(r) + S(r)\boldsymbol{v}_1(r),\end{aligned} \tag{4.6}$$

because $Q(r)S(r) = S(r)$ and

$$S(r)\Big|_{E^\perp(r)} = -\frac{dQ}{dr}(r)\Big|_{E^\perp(r)}$$

hold.

Similarly, $\boldsymbol{v}_2(r)$ satisfies

$$\begin{aligned}\frac{d\boldsymbol{v}_2}{dr} &= \frac{d}{dr}\{R(r)\boldsymbol{v}(r)\}\\ &= \frac{dR}{dr}(r)\boldsymbol{v}(r) + R(r)\frac{d\boldsymbol{v}}{dr}(r)\\ &= -\frac{dQ}{dr}(r)\boldsymbol{v}(r) + R(r)S(r)\boldsymbol{v}(r)\\ &= -\frac{dQ}{dr}(r)(\boldsymbol{v}_1(r)+\boldsymbol{v}_2(r)),\end{aligned} \tag{4.7}$$

because $\frac{dQ}{dr}(r) + \frac{dR}{dr}(r) = 0$ and $R(r)S(r) = 0$ hold.

(4.6) and (4.7) mean $\boldsymbol{v}_1(r) \equiv 0$ when $\boldsymbol{v}_1(0) = 0$. This shows that $\boldsymbol{v}(r) = \boldsymbol{v}_2(r) \in E^{\perp}(r)$, specially $\Pi(\boldsymbol{h})\boldsymbol{v}_0 = \boldsymbol{v}(1) \in E^{\perp}(\boldsymbol{h})$.

The continuity of $\Pi(\boldsymbol{h})$ and $\Pi^{-1}(\boldsymbol{h})$ is obvious. ∎

Fix $\rho_1 > 0$ and define $H(\widehat{\boldsymbol{h}}, \rho_1) = \{\boldsymbol{h} = (h_1, \cdots, h_N) \in \boldsymbol{R}^N\ ;\ \widehat{h}_j < h_j < \widehat{h}_j + \rho_1\}$, $\mathcal{M}(\widehat{\boldsymbol{h}}, \rho_1) = \{\Xi(l)P(z;\boldsymbol{h})\ ;\ l \in \boldsymbol{R}^1,\ \boldsymbol{h} \in H(\widehat{\boldsymbol{h}}, \rho_1)\}$. Then, from the construction of $\Pi(\boldsymbol{h})$, there exist a positive constant C_1 depending only on ρ_1 and independent of $\widehat{\boldsymbol{h}}$ with $\min \widehat{\boldsymbol{h}} > h^*$ for sufficiently large h^* such that for any $\boldsymbol{h} \in H(\widehat{\boldsymbol{h}}, \rho_1)$

$$\|\Pi(\boldsymbol{h})\|,\ \|\Pi^{-1}(\boldsymbol{h})\|,\ \|\frac{\partial}{\partial h_j}\Pi(\boldsymbol{h})\| \le C_1,$$

$$\|\Pi(\boldsymbol{h})\|_\infty,\ \|\Pi^{-1}(\boldsymbol{h})\|_\infty,\ \|\frac{\partial}{\partial h_j}\Pi(\boldsymbol{h})\|_\infty \le C_1,$$

hold, where $\|\cdot\|_\infty$ is an operator norm with respect to the sup-norm $\|\cdot\|_\infty$ on $\boldsymbol{R}^1$.

Let $A = L(\widehat{\boldsymbol{h}})$ and X^ω be the space with the norm $\|\cdot\|_\omega$ defined by the fractional power A^ω of A for $\omega \in [0,1)$. Hereafter, we fix ω in $\frac{3}{4} < \omega < 1$ such that X^ω is imbedded into $BU^1(\boldsymbol{R}^1)$(ref. [19]), where $BU^k(\boldsymbol{R}^1)$ is the space consisting of uniformly continuous and bounded functions on $\boldsymbol{R}^1$ up to their k th order derivatives.

Lemma 4.2. *There exists a neighborhood $U = U(\widehat{\boldsymbol{h}}, \rho_1)$ of $\mathcal{M}(\widehat{\boldsymbol{h}}, \rho_1)$ in X^ω such that any $\boldsymbol{v} \in U$ is represented by*

$$\boldsymbol{v} = \Xi(l)\{P(z;\boldsymbol{h}) + \Pi(\boldsymbol{h})\boldsymbol{w}\}$$

for $l \in \boldsymbol{R}^1$, $\boldsymbol{h} \in H(\widehat{\boldsymbol{h}}, \rho_1)$ and $\boldsymbol{w} \in E^{\perp}(\widehat{\boldsymbol{h}})$.

Proof. Fix $l_0 \in \boldsymbol{R}^1$, $\boldsymbol{h}_0 \in H(\widehat{\boldsymbol{h}}, \rho_1)$ arbitrarily and put $\boldsymbol{v}_0 = \Xi(l_0)P(\cdot;\boldsymbol{h}_0)$. We will show the existence of l, $\boldsymbol{h}$ and $\boldsymbol{w} \in E^{\perp}(\widehat{\boldsymbol{h}})$ near l_0, $\boldsymbol{h}_0$ and 0 for sufficiently small $\boldsymbol{v} \in X$ such that

$$\boldsymbol{v} + \boldsymbol{v}_0 = \Xi(l)\{P(\cdot,\boldsymbol{h}) + \Pi(\boldsymbol{h})\boldsymbol{w}\}.$$

Since $\Pi(\boldsymbol{h})$ is homeomorphic from $E^{\perp}(\widehat{\boldsymbol{h}})$ to $E^{\perp}(\boldsymbol{h})$, it suffices to show

$$\Xi(-l)(\boldsymbol{v} + \boldsymbol{v}_0) - P(\cdot;\boldsymbol{h}) \in E^{\perp}(\boldsymbol{h}).$$

This is equivalent to

$$\begin{aligned}0 &= \langle\ \Xi(-l)(\boldsymbol{v} + \boldsymbol{v}_0) - P(\cdot;\boldsymbol{h}), \boldsymbol{\phi}_j^*(\boldsymbol{h})\ \rangle_{L^2}\\ &= \langle\ \boldsymbol{v} + \boldsymbol{v}_0 - \Xi(l)P(\cdot;\boldsymbol{h}), \Xi(l)\boldsymbol{\phi}_j^*(\boldsymbol{h})\ \rangle_{L^2}\ \ (j = 0, 1, \cdots, N)\end{aligned}$$

Hence, defining

$$V(l, \boldsymbol{h}, \boldsymbol{v}) = \big(\langle\ \boldsymbol{v} + \boldsymbol{v}_0 - \Xi(l)P(\cdot;\boldsymbol{h}), \Xi(l)\boldsymbol{\phi}_j^*(\boldsymbol{h})\ \rangle_{L^2}\big)_{j=0,1,\cdots,N} \in \boldsymbol{R}^{N+1},$$

we can apply the implicit function theorem to the map V. First, we note $V(l_0, \boldsymbol{h}_0, 0) = 0$ holds. On the other hand, Proposition 4.2 shows

$$\langle\, P_z(z;\boldsymbol{h}), \boldsymbol{\phi}_j^*(\boldsymbol{h})\,\rangle_{L^2} = 1 + O(\delta(\boldsymbol{h})), \tag{4.8}$$

$$\langle\, \frac{\partial}{\partial h_k} P(z;\boldsymbol{h}), \boldsymbol{\phi}_j^*(\boldsymbol{h})\,\rangle_{L^2} = \begin{cases} -1 + O(\delta(\boldsymbol{h})) & (j \geq k), \\ O(\delta(\boldsymbol{h})) & (j < k). \end{cases} \tag{4.9}$$

By this and the fact $\Xi'(l) = -\Xi(l)\dfrac{\partial}{\partial z}$, we have

$$\begin{aligned} \frac{\partial V}{\partial l}(l_0, \boldsymbol{h}_0, 0) &= \left(\langle\, \Xi(l_0) P_z(\cdot;\boldsymbol{h}_0), \Xi(l_0)\boldsymbol{\phi}_j^*(\boldsymbol{h}_0)\,\rangle_{L^2}\right)_{j=0,1,\cdots,N} \\ &= \left(\langle\, P_z(\cdot;\boldsymbol{h}_0), \boldsymbol{\phi}_j^*(\boldsymbol{h}_0)\,\rangle_{L^2}\right)_{j=0,1,\cdots,N} \\ &= (1 + O(\delta(\boldsymbol{h})))_{j=0,1,\cdots,N}. \end{aligned}$$

Similarly, we have for $\boldsymbol{k} = (k_1, k_2, \cdots, k_N) \in \boldsymbol{R}^N$

$$\begin{aligned} \frac{\partial V}{\partial \boldsymbol{h}}(l_0, \boldsymbol{h}_0, 0)\boldsymbol{k} &= \left(\sum_{i=1}^{N} \langle\, -\Xi(l_0)\frac{\partial P}{\partial h_i}(\cdot, \boldsymbol{h}_0), \Xi(l_0)\boldsymbol{\phi}_j^*(\boldsymbol{h}_0)\,\rangle_{L^2} k_i\right)_{j=0,1,\cdots,N} \\ &= \left(-\sum_{i=1}^{N} \langle\, \frac{\partial P}{\partial h_i}(\cdot, \boldsymbol{h}_0), \boldsymbol{\phi}_j^*(\boldsymbol{h}_0)\,\rangle_{L^2} k_i\right)_{j=0,1,\cdots,N} \\ &= \left(\sum_{i=1}^{j} k_i + \sum_{i=1}^{N} O(\delta(\boldsymbol{h}))k_i\right). \end{aligned}$$

Thus, we find $\dfrac{\partial V}{\partial(l, \boldsymbol{h})}(l_0, \boldsymbol{h}_0, 0) = V_0 + O(\delta(\boldsymbol{h}))$, a square matrix of $N+1$ degree, where

$$V_0 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Since V_0 is a lower triangular matrix and invertible, $\dfrac{\partial V}{\partial(l, \boldsymbol{h})}(l_0, \boldsymbol{h}_0, 0)$ is invertible for sufficiently small $\delta(\boldsymbol{h})$ and the implicit function theorem shows there exist $l = l(\boldsymbol{v})$ and $\boldsymbol{h} = \boldsymbol{h}(\boldsymbol{v})$ for small $\boldsymbol{v}$ such that

$$V(l(\boldsymbol{v}), \boldsymbol{h}(\boldsymbol{v}), \boldsymbol{v}) = 0.$$

∎

We transform the equation (1.6) of $\boldsymbol{u}$ to that of $(\boldsymbol{w}, l, \boldsymbol{h})$ by

$$\boldsymbol{u}(t, z) = \Xi(l)\{P(z;\boldsymbol{h}) + \Pi(\boldsymbol{h})\boldsymbol{w}\}$$

for $l \in \boldsymbol{R}^1$, $\boldsymbol{h} \in H(\widehat{\boldsymbol{h}}, \rho_1)$ and $\boldsymbol{w} \in E^{\perp}(\widehat{\boldsymbol{h}})$. Since $\Xi'(l) = -\Xi(l)\dfrac{\partial}{\partial z}$ holds, we have

$$\begin{aligned} \boldsymbol{u}_t &= \dot{l}\Xi'(l)\{P(z;\boldsymbol{h}) + \Pi(\boldsymbol{h})\boldsymbol{w}\} + \Xi(l)\left(\frac{\partial}{\partial \boldsymbol{h}}\{P(z;\boldsymbol{h}) + \Pi(\boldsymbol{h})\boldsymbol{w}\}\dot{\boldsymbol{h}} + \Pi(\boldsymbol{h})\boldsymbol{w}_t\right) \\ &= \Xi(l)\left(-\dot{l}\frac{\partial}{\partial z}\{P(z;\boldsymbol{h}) + \Pi(\boldsymbol{h})\boldsymbol{w}\} + \frac{\partial}{\partial \boldsymbol{h}}\{P(z;\boldsymbol{h}) + \Pi(\boldsymbol{h})\boldsymbol{w}\}\dot{\boldsymbol{h}} + \Pi(\boldsymbol{h})\boldsymbol{w}_t\right) \end{aligned}$$

and

$$\begin{aligned}\mathcal{L}(\boldsymbol{u}) &= \mathcal{L}(\Xi(l)\{P(z;\boldsymbol{h})+\Pi(\boldsymbol{h})\boldsymbol{w}\}) \\ &= \Xi(l)\mathcal{L}(P(z;\boldsymbol{h})+\Pi(\boldsymbol{h})\boldsymbol{w}).\end{aligned}$$

Hence, it follows that

$$-\imath\frac{\partial}{\partial z}\{P(z;\boldsymbol{h})+\Pi(\boldsymbol{h})\boldsymbol{w}\}+\frac{\partial}{\partial \boldsymbol{h}}\{P(z;\boldsymbol{h})+\Pi(\boldsymbol{h})\boldsymbol{w}\}\dot{\boldsymbol{h}}+\Pi(\boldsymbol{h})\boldsymbol{w}_t=\mathcal{L}\left(P(z;\boldsymbol{h})+\Pi(\boldsymbol{h})\boldsymbol{w}\right)$$

and we have

$$\begin{aligned}Q(\boldsymbol{h})&\left[-\imath\frac{\partial}{\partial z}\{P(z;\boldsymbol{h})+\Pi(\boldsymbol{h})\boldsymbol{w}\}+\frac{\partial}{\partial \boldsymbol{h}}\{P(z;\boldsymbol{h})+\Pi(\boldsymbol{h})\boldsymbol{w}\}\dot{\boldsymbol{h}}\right] \\ &= Q(\boldsymbol{h})\mathcal{L}(P(z;\boldsymbol{h})+\Pi(\boldsymbol{h})\boldsymbol{w}),\end{aligned} \tag{4.10}$$

$$\begin{aligned}\Pi^{-1}(\boldsymbol{h})R(\boldsymbol{h})&\left[-\imath\frac{\partial}{\partial z}\{P(z;\boldsymbol{h})+\Pi(\boldsymbol{h})\boldsymbol{w}\}+\frac{\partial}{\partial \boldsymbol{h}}\{P(z;\boldsymbol{h})+\Pi(\boldsymbol{h})\boldsymbol{w}\}\dot{\boldsymbol{h}}\right]+\boldsymbol{w}_t \\ &= \Pi^{-1}(\boldsymbol{h})R(\boldsymbol{h})\mathcal{L}(P(z;\boldsymbol{h})+\Pi(\boldsymbol{h})\boldsymbol{w}).\end{aligned} \tag{4.11}$$

Let $\rho_2>0$ and $C_2>0$ be constants such that if $\|\boldsymbol{w}\|_\omega<\rho_2$ and $\boldsymbol{h}\in H(\widehat{\boldsymbol{h}},\rho_1)$, then

$$|\mathcal{L}(P(z;\boldsymbol{h})+\Pi(\boldsymbol{h})\boldsymbol{w})-\mathcal{L}(P(z;\boldsymbol{h}))-L(\boldsymbol{h})\Pi(\boldsymbol{h})\boldsymbol{w}|\le C_2|\boldsymbol{w}|^2 \tag{4.12}$$

holds. We note here that ρ_2 is taken to be independent of $\widehat{\boldsymbol{h}}$ and depending only on ρ_1.

Put

$$W(\widehat{\boldsymbol{h}},\rho_1,D_1)=\{\boldsymbol{w}(\cdot)\in C\left(H(\widehat{\boldsymbol{h}},\rho_1);E^{\perp}(\widehat{\boldsymbol{h}})\cap X^{\omega}\right)\ ;\ \|\boldsymbol{w}(\boldsymbol{h})\|_\omega<D_1\delta(\boldsymbol{h})\}.$$

We determine D_1 later but suppose h^* is large enough so as to $D_1\delta(\boldsymbol{h})<\rho_2$ for $\boldsymbol{h}\in H(\widehat{\boldsymbol{h}},\rho_1)$ with $\min\widehat{\boldsymbol{h}}>h^*$.

First, we consider (4.10). It follows that if $\|\boldsymbol{w}\|_\omega<D_1\delta(\boldsymbol{h})$, then we have from (4.12)

$$\begin{aligned}&\langle\ \mathcal{L}(P(z;\boldsymbol{h})+\Pi(\boldsymbol{h})\boldsymbol{w}),\boldsymbol{\phi}_j^*(\boldsymbol{h})\ \rangle_{L^2} \\ &=\langle\ \mathcal{L}(P(z;\boldsymbol{h}))+L(\boldsymbol{h})\Pi(\boldsymbol{h})\boldsymbol{w}+O(\delta^2),\boldsymbol{\phi}_j^*(\boldsymbol{h})\ \rangle_{L^2} \\ &=\langle\ \mathcal{L}(P(z;\boldsymbol{h})),\boldsymbol{\phi}_j^*(\boldsymbol{h})\ \rangle_{L^2}+\langle\ \Pi(\boldsymbol{h})\boldsymbol{w},L^*(\boldsymbol{h})\boldsymbol{\phi}_j^*(\boldsymbol{h})\ \rangle_{L^2}+O(\delta^2) \\ &=\langle\ \mathcal{L}(P(z;\boldsymbol{h})),\boldsymbol{\phi}_j^*(\boldsymbol{h})\ \rangle_{L^2}+O(\delta^2) \\ &=H_j(\boldsymbol{h})+O(\delta^2),\end{aligned} \tag{4.13}$$

where $\delta=\delta(\boldsymbol{h})$. Here, we used the fact $L^*(\boldsymbol{h})\boldsymbol{\phi}_j^*(\boldsymbol{h})=O(\delta(\boldsymbol{h}))$.

LEMMA 4.3.
Let $\widetilde{\Pi}(\boldsymbol{h})\boldsymbol{w}=\dfrac{\partial}{\partial z}\{\Pi(\boldsymbol{h})\boldsymbol{w}\}$ *for* $\boldsymbol{w}\in X^\omega$. *Then,*

$$\|\widetilde{\Pi}(\boldsymbol{h})\boldsymbol{w}\|\le C\|\boldsymbol{w}\|_\omega$$

holds.

Proof. Let $\boldsymbol{v}(r)$ be the solution of (4.5) with $\boldsymbol{v}_0=\boldsymbol{w}$. We shall show $\dfrac{\partial}{\partial z}v(r)$ is estimated at $\|\dfrac{\partial}{\partial z}v(r)\|\le C\|\boldsymbol{w}\|_\omega$.

Let $\widetilde{\boldsymbol{v}}(r)=\dfrac{\partial}{\partial z}v(r)$. Differentiating the both sides of (4.5) by z, we have

$$\frac{d\widetilde{\boldsymbol{v}}}{dr}=\widetilde{S}(r;\boldsymbol{h})\boldsymbol{v} \tag{4.14}$$

with $\widetilde{\boldsymbol{v}}(0) = \boldsymbol{w}_z$. Here,

$$\widetilde{S}(r;\boldsymbol{h})\boldsymbol{v} = \frac{\partial}{\partial z}S(r;\boldsymbol{h})\boldsymbol{v}$$
$$= -\sum_{j,k=0}^{N}(h_j - \widehat{h}_j)\,\langle\, \boldsymbol{v}, \frac{\partial}{\partial h_j}\boldsymbol{\phi}_k^*(\Theta(r))\,\rangle_{L^2}\,\frac{\partial}{\partial z}\boldsymbol{\phi}_k(\Theta(r)),$$

which is a bounded operator for $\boldsymbol{v} \in X$. Thus, we have from (4.14)

$$\widetilde{\boldsymbol{v}}(r) = \boldsymbol{w}_z + \int_0^r \widetilde{S}(r;\boldsymbol{h})\boldsymbol{v}(r)dr,$$

especially,

$$\widetilde{\Pi}(\boldsymbol{h})\boldsymbol{w} = \widetilde{\boldsymbol{v}}(1) = \boldsymbol{w}_z + \int_0^1 \widetilde{S}(r;\boldsymbol{h})\boldsymbol{v}(r)dr.$$

This gives the proof.

∎

Since

$$Q(\boldsymbol{h})\boldsymbol{v} = \sum_{j=0}^{N}\langle\, \boldsymbol{v}, \boldsymbol{\phi}_j^*(\boldsymbol{h})\,\rangle_{L^2}\,\boldsymbol{\phi}_j(\boldsymbol{h})$$

for $\boldsymbol{v} \in X$, (4.10) yields from (4.8), (4.9), (4.13) and Lemma 4.3,

$$-\dot{l}(1+O(\delta)) + \sum_{k=1}^{j}(-1+O(\delta))\dot{h}_k = H_j(\boldsymbol{h}) + O(\delta^2)\ (j = 1, 2, \cdots, N), \tag{4.15}$$

$$-\dot{l}(1+O(\delta)) = H_0(\boldsymbol{h}) + O(\delta^2) \tag{4.16}$$

if $\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{h}) \in W(\widehat{\boldsymbol{h}}, \rho_1, D_1)$, where $\delta = \delta(\boldsymbol{h})$. These and (4.10) imply that there exist functions $\widetilde{H}_j = \widetilde{H}(\boldsymbol{h}, \boldsymbol{w})$ $(j = 0, 1, \cdots, N)$ such that

$$\dot{l} = \widetilde{H}_0(\boldsymbol{h}, \boldsymbol{w}) = -H_0(\boldsymbol{h}) + O(\delta^2(\boldsymbol{h})), \tag{4.17}$$

$$\begin{aligned} \dot{h}_j &= \widetilde{H}_j(\boldsymbol{h}, \boldsymbol{w}) \\ &= H_{j-1}(\boldsymbol{h}) - H_j(\boldsymbol{h}) + O(\delta^2(\boldsymbol{h}))\ (j = 1, 2, \cdots, N). \end{aligned} \tag{4.18}$$

Especially,

$$\dot{l},\ \dot{h}_j = O(\delta(\boldsymbol{h})) \tag{4.19}$$

hold for $\boldsymbol{w} \in W(\widehat{\boldsymbol{h}}, \rho_1, D_1)$. Similarly, it follows from (4.11) and (4.12) that

$$\boldsymbol{w}_t = A(\boldsymbol{h})\boldsymbol{w} + \widetilde{G}(\boldsymbol{h}, \boldsymbol{w}) \tag{4.20}$$

with $\|\widetilde{G}\| = O(\delta(\boldsymbol{h}))$ for $\boldsymbol{h} \in H(\widehat{\boldsymbol{h}}, \rho_1,)$ and $\boldsymbol{w} \in W(\widehat{\boldsymbol{h}}, \rho_1, D_1)$, where

$$\begin{aligned} A(\boldsymbol{h}) &= \Pi^{-1}(\boldsymbol{h})L(\boldsymbol{h})\Pi(\boldsymbol{h}), \\ \widetilde{G}(\boldsymbol{h}, \boldsymbol{w}) &= \Pi^{-1}(\boldsymbol{h})R(\boldsymbol{h})\,[\mathcal{L}(P(z;\boldsymbol{h})) + L_2(\boldsymbol{w}, \boldsymbol{w}) \\ &\qquad + \widetilde{H}_0\frac{\partial}{\partial z}\{P(z;\boldsymbol{h}) + \Pi(\boldsymbol{h})\boldsymbol{w}\} - \frac{\partial}{\partial \boldsymbol{h}}\{P(z;\boldsymbol{h}) + \Pi(\boldsymbol{h})\boldsymbol{w}\}\widetilde{\boldsymbol{H}}\Big], \\ L_2(\boldsymbol{w}, \boldsymbol{w}) &= L_2(\boldsymbol{h}, \boldsymbol{w})(\boldsymbol{w}, \boldsymbol{w}) \\ &= \mathcal{L}(P(z;\boldsymbol{h}) + \Pi(\boldsymbol{h})\boldsymbol{w}) - \mathcal{L}(P(z;\boldsymbol{h})) - L(\boldsymbol{h})\Pi(\boldsymbol{h})\boldsymbol{w}, \\ \widetilde{\boldsymbol{H}} &= \widetilde{\boldsymbol{H}}(\boldsymbol{h}, \boldsymbol{w}) \\ &= (\widetilde{H}_1, \widetilde{H}_2, \cdots, \widetilde{H}_N). \end{aligned}$$

LEMMA 4.4.

$$\|A(\boldsymbol{h}_1)\boldsymbol{w} - A(\boldsymbol{h}_2)\boldsymbol{w}\| \le C|\boldsymbol{h}_1 - \boldsymbol{h}_2| \cdot \|\boldsymbol{w}\|$$

holds for $\boldsymbol{h}_j \in H(\widehat{\boldsymbol{h}}, \rho_1)$ $(j = 1, 2)$ *and* $\boldsymbol{w} \in \left\{H^2(\boldsymbol{R}^1)\right\}^n$.

Proof. From the definition of $A(\boldsymbol{h}_j)$, we have

$$\begin{aligned}
A(\boldsymbol{h}_1)\boldsymbol{w} - A(\boldsymbol{h}_2)\boldsymbol{w} &= \Pi^{-1}(\boldsymbol{h}_1)L(\boldsymbol{h}_1)\Pi(\boldsymbol{h}_1)\boldsymbol{w} - \Pi^{-1}(\boldsymbol{h}_2)L(\boldsymbol{h}_2)\Pi(\boldsymbol{h}_2)\boldsymbol{w} \\
&= \Pi^{-1}(\boldsymbol{h}_1)L(\boldsymbol{h}_1)\Pi(\boldsymbol{h}_2)\boldsymbol{w} - \Pi^{-1}(\boldsymbol{h}_2)L(\boldsymbol{h}_1)\Pi(\boldsymbol{h}_2)\boldsymbol{w} \qquad (4.21)\\
&\quad + \Pi^{-1}(\boldsymbol{h}_1)L(\boldsymbol{h}_1)\{\Pi(\boldsymbol{h}_1)\boldsymbol{w} - \Pi(\boldsymbol{h}_2)\boldsymbol{w}\} \qquad (4.22)\\
&\quad + \Pi^{-1}(\boldsymbol{h}_2)L(\boldsymbol{h}_1)\Pi(\boldsymbol{h}_2)\boldsymbol{w} - \Pi^{-1}(\boldsymbol{h}_2)L(\boldsymbol{h}_2)\Pi(\boldsymbol{h}_2)\boldsymbol{w} \qquad (4.23)
\end{aligned}$$

for $\boldsymbol{w} \in \left\{H^2(\boldsymbol{R}^1)\right\}^n$. We shall estimate (4.21), (4.22) and (4.23).

(4.23) is easily estimated by

$$\begin{aligned}
\|L(\boldsymbol{h}_1)\boldsymbol{v} - L(\boldsymbol{h}_2)\boldsymbol{v}\| &= \|F'(P(z;\boldsymbol{h}_1))\boldsymbol{v} - F'(P(z;\boldsymbol{h}_2))\boldsymbol{v}\| \qquad (4.24)\\
&\le C \sup_z |F'(P(z;\boldsymbol{h}_1)) - F'(P(z;\boldsymbol{h}_2))| \cdot \|\boldsymbol{v}\| \\
&\le C|\boldsymbol{h}_1 - \boldsymbol{h}_2| \cdot \|\boldsymbol{v}\|
\end{aligned}$$

becuase $\sup_z |P(z;\boldsymbol{h})|,\ \sup_z |P_z(z;\boldsymbol{h})| \le C$ holds uniformly for any $\boldsymbol{h} \in \boldsymbol{R}^N$, where $\boldsymbol{v} = \Pi(\boldsymbol{h}_2)\boldsymbol{w}$.

Next, we consider (4.22). Let $\boldsymbol{v}(r;\boldsymbol{h}_j)$ be the solution of (4.5) with $\boldsymbol{v}(0;\boldsymbol{h}_j) = \boldsymbol{w}$. By the definition of $\Pi(\boldsymbol{h}_j)$ and $\boldsymbol{v}(0;\boldsymbol{h}_j) = \Pi(\widehat{\boldsymbol{h}})\boldsymbol{w}$, we have

$$\begin{aligned}
\Pi(\boldsymbol{h}_1)\boldsymbol{w} - \Pi(\boldsymbol{h}_2)\boldsymbol{w} &= \{\Pi(\boldsymbol{h}_1)\boldsymbol{w} - \Pi(\widehat{\boldsymbol{h}})\boldsymbol{w}\} - \{\Pi(\boldsymbol{h}_2)\boldsymbol{w} - \Pi(\widehat{\boldsymbol{h}})\boldsymbol{w}\} \\
&= \{\boldsymbol{v}(1;\boldsymbol{h}_1) - \boldsymbol{v}(0;\boldsymbol{h}_1)\} - \{\boldsymbol{v}(1;\boldsymbol{h}_2) - \boldsymbol{v}(0;\boldsymbol{h}_2)\} \\
&= \int_0^1 \frac{d\boldsymbol{v}}{dr}(r;\boldsymbol{h}_1)dr - \int_0^1 \frac{d\boldsymbol{v}}{dr}(r;\boldsymbol{h}_2)dr \\
&= \int_0^1 \{S(r;\boldsymbol{h}_1)\boldsymbol{v}(r;\boldsymbol{h}_1) - S(r;\boldsymbol{h}_2)\boldsymbol{v}(r;\boldsymbol{h}_2)\}dr.
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
&\|\Pi^{-1}(\boldsymbol{h}_1)L(\boldsymbol{h}_1)\{\Pi(\boldsymbol{h}_1)\boldsymbol{w} - \Pi(\boldsymbol{h}_2)\boldsymbol{w}\}\| \qquad (4.25)\\
&\le C\int_0^1 \|L(\boldsymbol{h}_1)\{S(r;\boldsymbol{h}_1)\boldsymbol{v}(r;\boldsymbol{h}_1) - S(r;\boldsymbol{h}_2)\boldsymbol{v}(r;\boldsymbol{h}_2)\}\|dr \\
&\le C\int_0^1 \|L(\boldsymbol{h}_1)S(r;\boldsymbol{h}_1)\boldsymbol{v}(r;\boldsymbol{h}_1) - L(\boldsymbol{h}_1)S(r;\boldsymbol{h}_2)\boldsymbol{v}(r;\boldsymbol{h}_1)\|dr \\
&\qquad + C\int_0^1 \|L(\boldsymbol{h}_1)S(r;\boldsymbol{h}_2)\{\boldsymbol{v}(r;\boldsymbol{h}_1) - \boldsymbol{v}(r;\boldsymbol{h}_2)\}\|dr \\
&\le C\int_0^1 \|L(\boldsymbol{h}_1)S(r;\boldsymbol{h}_1) - L(\boldsymbol{h}_1)S(r;\boldsymbol{h}_2)\| \cdot \|\boldsymbol{v}(r;\boldsymbol{h}_1)\|dr \\
&\qquad + C\int_0^1 \|L(\boldsymbol{h}_1)S(r;\boldsymbol{h}_2)\| \cdot \|\boldsymbol{v}(r;\boldsymbol{h}_1) - \boldsymbol{v}(r;\boldsymbol{h}_2)\|dr \\
&\le C|\boldsymbol{h}_1 - \boldsymbol{h}_2| \cdot \|\boldsymbol{w}\|
\end{aligned}$$

for $\boldsymbol{h}_j \in H(\widehat{\boldsymbol{h}}, \rho_1)$ because $L(\boldsymbol{h}_1)S(r;\boldsymbol{h})$ is a bounded operator on X depending smoothly on $\boldsymbol{h} \in H(\widehat{\boldsymbol{h}}, \rho_1)$, $r \in [0,\ 1]$ and $\boldsymbol{v}(r;\boldsymbol{h})$ is also a function depending smoothly on $\boldsymbol{h} \in H(\widehat{\boldsymbol{h}}, \rho_1)$, $r \in [0,\ 1]$.

Finally, we estimate (4.21). Define the operator $\widehat{\Pi}(r;\boldsymbol{h})$ by $\widehat{\Pi}(r;\boldsymbol{h})\boldsymbol{v}_0 = \boldsymbol{v}(r;\boldsymbol{h})$, where $\boldsymbol{v}(r;\boldsymbol{h})$ is a solution of (4.5).

Proposition 4.3.

$$\frac{d}{dr}\left\{\widehat{\Pi}^{-1}(r;\boldsymbol{h})\boldsymbol{v}_0\right\} = -\widehat{\Pi}^{-1}(r;\boldsymbol{h})S(r;\boldsymbol{h})\boldsymbol{v}_0$$

holds for $\boldsymbol{h} \in H(\widehat{\boldsymbol{h}}, \rho_1)$, $\boldsymbol{v}_0 \in X$ *and* $r \in [0,\ 1]$.

Proof. Let $\boldsymbol{u}(r) = \widehat{\Pi}^{-1}(r;\boldsymbol{h})\boldsymbol{v}_0$. Then, we have

$$\boldsymbol{v}_0 = \widehat{\Pi}(r;\boldsymbol{h})\boldsymbol{u}(r)$$

and therefore

$$\begin{aligned}
0 &= \frac{d\widehat{\Pi}}{dr}(r;\boldsymbol{h})\boldsymbol{u}(r) + \widehat{\Pi}(r;\boldsymbol{h})\frac{d\boldsymbol{u}}{dr}(r) \\
&= S(r;\boldsymbol{h})\widehat{\Pi}(r;\boldsymbol{h})\boldsymbol{u}(r) + \widehat{\Pi}(r;\boldsymbol{h})\frac{d\boldsymbol{u}}{dr}(r) \\
&= S(r;\boldsymbol{h})\boldsymbol{v}_0 + \widehat{\Pi}(r;\boldsymbol{h})\frac{d\boldsymbol{u}}{dr}(r)
\end{aligned}$$

holds. This implies the proof. ∎

Similarly to the case of (4.22), we have from Proposition 4.3

$$\begin{aligned}
&\Pi^{-1}(\boldsymbol{h}_1)L(\boldsymbol{h}_1)\boldsymbol{v} - \Pi^{-1}(\boldsymbol{h}_2)L(\boldsymbol{h}_1)\boldsymbol{v} \\
&= \{\Pi^{-1}(\boldsymbol{h}_1)L(\boldsymbol{h}_1)\boldsymbol{v} - \Pi^{-1}(\widehat{\boldsymbol{h}})L(\boldsymbol{h}_1)\boldsymbol{v}\} + \{\Pi^{-1}(\widehat{\boldsymbol{h}})L(\boldsymbol{h}_1)\boldsymbol{v} - \Pi^{-1}(\boldsymbol{h}_2)L(\boldsymbol{h}_1)\boldsymbol{v}\} \\
&= \int_0^1 \frac{d}{dr}\left\{\widehat{\Pi}^{-1}(r;\boldsymbol{h}_1)L(\boldsymbol{h}_1)\boldsymbol{v}\right\}dr - \int_0^1 \frac{d}{dr}\left\{\widehat{\Pi}^{-1}(r;\boldsymbol{h}_2)L(\boldsymbol{h}_1)\boldsymbol{v}\right\}dr \\
&= -\int_0^1 \widehat{\Pi}^{-1}(r;\boldsymbol{h}_1)S(r;\boldsymbol{h}_1)L(\boldsymbol{h}_1)\boldsymbol{v}dr + \int_0^1 \widehat{\Pi}^{-1}(r;\boldsymbol{h}_2)S(r;\boldsymbol{h}_2)L(\boldsymbol{h}_1)\boldsymbol{v}dr,
\end{aligned}$$

where $\boldsymbol{v} = \Pi(\boldsymbol{h}_2)\boldsymbol{w}$. Hence, it follows that

$$\begin{aligned}
&\|\Pi^{-1}(\boldsymbol{h}_1)L(\boldsymbol{h}_1)\boldsymbol{v} - \Pi^{-1}(\boldsymbol{h}_2)L(\boldsymbol{h}_1)\boldsymbol{v}\| \\
&\le \int_0^1 \|\widehat{\Pi}^{-1}(r;\boldsymbol{h}_1)S(r;\boldsymbol{h}_1)L(\boldsymbol{h}_1) - \widehat{\Pi}^{-1}(r;\boldsymbol{h}_2)S(r;\boldsymbol{h}_2)L(\boldsymbol{h}_1)\|dr\|\boldsymbol{v}\| \\
&\le C|\boldsymbol{h}_1 - \boldsymbol{h}_2| \cdot \|\boldsymbol{v}\| \\
&\le C|\boldsymbol{h}_1 - \boldsymbol{h}_2| \cdot \|\boldsymbol{w}\|
\end{aligned} \tag{4.26}$$

because the smooth dependence of $\widehat{\Pi}^{-1}(r;\boldsymbol{h})S(r;\boldsymbol{h})L(\boldsymbol{h}_1)$ on the argument $\boldsymbol{h}$.

(4.24), (4.25) and (4.26) show the proof. ∎

Remark 4.1. *It is shown that*

$$\|A(\boldsymbol{h})\boldsymbol{w} - A\boldsymbol{w}\| \le C|\boldsymbol{h} - \widehat{\boldsymbol{h}}| \cdot \|\boldsymbol{w}\| \tag{4.27}$$

for $\boldsymbol{h} \in H(\widehat{\boldsymbol{h}}, \rho_1)$ *and* $\boldsymbol{w} \in X^\omega$ *in quite a similar manner to this lemma but we use*

$$\begin{aligned}
A(\boldsymbol{h})\boldsymbol{w} - A\boldsymbol{w} &= \Pi^{-1}(\boldsymbol{h})L(\boldsymbol{h})\Pi(\boldsymbol{h})\boldsymbol{w} - L(\widehat{\boldsymbol{h}})\boldsymbol{w} \\
&= \Pi^{-1}(\boldsymbol{h})L(\boldsymbol{h})\boldsymbol{w} - L(\boldsymbol{h})\boldsymbol{w} \\
&\quad + \Pi^{-1}(\boldsymbol{h})L(\boldsymbol{h})\{\Pi(\boldsymbol{h})\boldsymbol{w} - \boldsymbol{w}\} \\
&\quad + L(\boldsymbol{h})\boldsymbol{w} - L(\widehat{\boldsymbol{h}})\boldsymbol{w}
\end{aligned}$$

Here, we note there exist $C_3 > 0$ (independent of D_1) and $C_4 = C_4(D_1) > 0$ (dependent on D_1) in (4.18) and (4.20) such that

$$|\widetilde{\boldsymbol{H}}(\boldsymbol{h},\boldsymbol{w})| \le C_3\delta(\boldsymbol{h}), \tag{4.28}$$

$$|\widetilde{\boldsymbol{H}}(\boldsymbol{h},\boldsymbol{w}) - \widetilde{\boldsymbol{H}}(\boldsymbol{k},\boldsymbol{v})| \le C_4\delta(\boldsymbol{h})\{|\boldsymbol{h}-\boldsymbol{k}| + \|\boldsymbol{w}-\boldsymbol{v}\|_\omega\}, \tag{4.29}$$

$$\|\widetilde{G}(\boldsymbol{h},\boldsymbol{w})\| \le C_3\delta(\boldsymbol{h})\{1 + (D_1 + D_1^2)\delta(\boldsymbol{h})\}, \tag{4.30}$$

$$\|\widetilde{G}(\boldsymbol{h},\boldsymbol{w}) - \widetilde{G}(\boldsymbol{k},\boldsymbol{v})\| \le C_4\{\delta(\boldsymbol{h}) + \delta(\boldsymbol{k})\}\{|\boldsymbol{h}-\boldsymbol{k}| + \|\boldsymbol{w}-\boldsymbol{v}\|_\omega\} \tag{4.31}$$

hold for $\boldsymbol{h},\ \boldsymbol{k} \in H(\widehat{\boldsymbol{h}},\rho_1)$ and $\boldsymbol{w},\ \boldsymbol{v} \in X^\omega$ with $\|\boldsymbol{w}\|_\omega,\ \|\boldsymbol{v}\|_\omega \le D_1\delta(\boldsymbol{h})$. We extend $\delta(\boldsymbol{h})$ for $\boldsymbol{h} \notin H(\widehat{\boldsymbol{h}},\rho_1)$ such that $\delta(\boldsymbol{h}) \le \delta^* = \delta(h^*)$ holds for any $\boldsymbol{h} \in \boldsymbol{R}^N$ and also extend $\widetilde{\boldsymbol{H}}$ and $\widetilde{G}$ appropriately to the outside of $H(\widehat{\boldsymbol{h}},\rho_1)$ such that (4.28) $\sim$ (4.31) hold for any $\boldsymbol{h},\ \boldsymbol{k} \in \boldsymbol{R}^N$.

We shall construct an attractive invariant manifold of

$$\begin{cases} \boldsymbol{h}_t &= \widetilde{\boldsymbol{H}}(\boldsymbol{h},\boldsymbol{w}), \\ \boldsymbol{w}_t &= A(\boldsymbol{h})\boldsymbol{w} + \widetilde{G}(\boldsymbol{h},\boldsymbol{w}) \end{cases} \tag{4.32}$$

for $\boldsymbol{h} \in \boldsymbol{R}^N$ and $\boldsymbol{w} \in W(\widehat{\boldsymbol{h}},\rho_1,D_1)$. Since the resolvent of $A(\boldsymbol{h})$ satisfies

$$\|(\lambda - A(\boldsymbol{h}))^{-1}\| \le \frac{C}{|\lambda + \rho_0|} \tag{4.33}$$

uniformly for $\boldsymbol{h} \in H(\widehat{\boldsymbol{h}},\rho_1)$ on $E^\perp(\widehat{\boldsymbol{h}})$, we may assume $A(\boldsymbol{h})$ is extended for $\boldsymbol{h} \notin H(\widehat{\boldsymbol{h}},\rho_1)$ such that (4.33) holds for $\boldsymbol{h} \in \boldsymbol{R}^N$. We also assume $A(\boldsymbol{h})$ is extended for $\boldsymbol{h} \notin H(\widehat{\boldsymbol{h}},\rho_1)$ such that Lemma 4.4 and (4.27) hold for $\boldsymbol{h} \in \boldsymbol{R}^N$.

Define $\delta(\boldsymbol{h},\boldsymbol{k}) = \delta(\boldsymbol{h}) + \delta(\boldsymbol{k})$ for $\boldsymbol{h},\ \boldsymbol{k} \in \boldsymbol{R}^N$. Note that $\delta(\boldsymbol{h},\boldsymbol{k}) \le 2\delta^*$ holds. Let

$$\begin{aligned} &\widetilde{W}(D_1,D_2) \\ &= \{\boldsymbol{w} \in C\left(\boldsymbol{R}^N; E^\perp(\widehat{\boldsymbol{h}}) \cap X^\omega\right)\ ;\ \|\boldsymbol{w}(\boldsymbol{h})\|_\omega \le D_1\delta(\boldsymbol{h}), \\ &\|\boldsymbol{w}(\boldsymbol{h}) - \boldsymbol{w}(\boldsymbol{k})\|_\omega \le D_2\delta(\boldsymbol{h},\boldsymbol{k})|\boldsymbol{h}-\boldsymbol{k}|\ for\ \boldsymbol{h},\ \boldsymbol{k} \in \boldsymbol{R}^N\} \end{aligned}$$

for a positive constant D_2. For $\sigma \in \widetilde{W}(D_1,D_2)$, define $\boldsymbol{h}(t) = \boldsymbol{h}(t;\xi,\sigma)$ by the solution of

$$\begin{cases} \boldsymbol{h}_t &= \widetilde{\boldsymbol{H}}(\boldsymbol{h},\sigma(\boldsymbol{h})), \\ \boldsymbol{h}(0) &= \xi \in \boldsymbol{R}^N \end{cases} \tag{4.34}$$

and define $T(t,s) = T(t,s;\boldsymbol{h}(\cdot))$ by the evolution operator of

$$\boldsymbol{w}_t = A(\boldsymbol{h}(t))\boldsymbol{w}. \tag{4.35}$$

LEMMA 4.5. *There exist positive constants δ_0, C_5 and γ_1 independent of D_1 such that if $\boldsymbol{h}(t) \in C^1(\boldsymbol{R};\boldsymbol{R}^N)$ satisfies $|\boldsymbol{h}_t| \le \delta_0$, then $\|T(t,s;\boldsymbol{h}(\cdot))\boldsymbol{w}\|_\omega \le C_5\max\{(t-s)^{-\omega},1\}e^{-\gamma_1(t-s)}\|\boldsymbol{w}\|$ holds for $\boldsymbol{w} \in E^\perp(\widehat{\boldsymbol{h}})$.*

Proof. (4.33), (4.27) and e.g. Theorem 7.4.2 of [19] directly give the proof. ∎

Hereafter, we take h^* large enough that $C_3\delta^* \le \delta_0$.

For $\boldsymbol{h}(\cdot) \in C^1(\boldsymbol{R};\boldsymbol{R}^N)$ with $|\boldsymbol{h}_t| \le C_3\delta(\boldsymbol{h}(t)) \le \delta_0$, consider a bounded solution of

$$\boldsymbol{w}_t = A(\boldsymbol{h}(t))\boldsymbol{w} + \widetilde{G}(\boldsymbol{h}(t),\boldsymbol{w}),\ -\infty < t < +\infty. \tag{4.36}$$

LEMMA 4.6. *There exists a constant D_1 such that a bounded solution of (4.36), say $\boldsymbol{w}(t;\boldsymbol{h}(\cdot))$, uniquely exists satisfying*

$$\|\boldsymbol{w}(t;\boldsymbol{h}(\cdot))\|_\omega \le D_1\delta(\boldsymbol{h}(t)).$$

Proof. Let $\boldsymbol{v}(t)$ be a function satisfying

$$\|\boldsymbol{v}(t)\|_\omega \le D_1\delta(\boldsymbol{h}(t)),\ -\infty < t < +\infty$$

and consider a bounded solution of

$$\boldsymbol{w}_t = A(\boldsymbol{h}(t))\boldsymbol{w} + \widetilde{G}(\boldsymbol{h}(t), \boldsymbol{v}(t)). \tag{4.37}$$

Solutions of (4.37) is represented as

$$\boldsymbol{w}(t) = T(t,s)\boldsymbol{w}(s) + \int_s^t T(t,\eta)\widetilde{G}(\boldsymbol{h}(\eta), \boldsymbol{v}(\eta))d\eta,$$

where $T(t,s) = T(t,s;\boldsymbol{h}(\cdot))$. Since $\|\boldsymbol{w}(t)\|_\omega$ is bounded as $t \to -\infty$, $\boldsymbol{w}(t)$ satisfies

$$\boldsymbol{w}(t) = \int_{-\infty}^t T(t,s)\widetilde{G}(\boldsymbol{h}(s), \boldsymbol{v}(s))ds. \tag{4.38}$$

Let $W(t;\boldsymbol{v}(\cdot))$ be the right hand side of (4.38). Then we have from Lemma 4.5

$$\begin{aligned}
&\|W(t;\boldsymbol{v}(\cdot))\|_\omega \\
&\le \int_{-\infty}^t C_5 \max\{(t-s)^{-\omega}, 1\}e^{-\gamma_1(t-s)}\|\widetilde{G}(\boldsymbol{h}(s), \boldsymbol{v}(s)\|ds \\
&\le \int_{-\infty}^t C_5 \max\{(t-s)^{-\omega}, 1\}e^{-\gamma_1(t-s)}C_3\delta(\boldsymbol{h}(s))\{1 + (D_1 + D_1^2)\delta(\boldsymbol{h}(s))\}ds \\
&\le \int_{-\infty}^t C \max\{(t-s)^{-\omega}, 1\}e^{-\gamma_1(t-s)}\delta(\boldsymbol{h}(s))ds \cdot \{1 + (D_1 + D_1^2)\delta^*\} \\
&= C\int_0^\infty \max\{s^{-\omega}, 1\}e^{-\gamma_1 s}\delta(\boldsymbol{h}(t-s))ds \cdot \{1 + (D_1 + D_1^2)\delta^*\} \\
&\le C\int_0^\infty \max\{s^{-\omega}, 1\}e^{-\gamma_1' s}\{e^{-(\gamma_1-\gamma_1')s}\delta(\boldsymbol{h}(t-s))\}ds \cdot \{1 + (D_1 + D_1^2)\delta^*\} \\
&\le C\int_0^\infty \max\{s^{-\omega}, 1\}e^{-\gamma_1' s}ds \cdot \delta(\boldsymbol{h}(t))\{1 + (D_1 + D_1^2)\delta^*\} \\
&\le C\delta(\boldsymbol{h}(t))\{1 + (D_1 + D_1^2)\delta^*\}
\end{aligned} \tag{4.39}$$

for a positive constant γ_1' with $0 < \gamma_1' < \gamma_1$. Here, we used the monotone decrement of $e^{-(\gamma_1-\gamma_1')s}\delta(\boldsymbol{h}(t-s))$ with respect to s, which is due to $\frac{d}{ds}\delta(\boldsymbol{h}(t-s)) = O(\delta^2(\boldsymbol{h}(t-s))) \le \delta(\boldsymbol{h}(t-s))O(\delta^*)$. Hence, we take D_1 and h^* so large that

$$C\{1 + (D_1 + D_1^2)\delta^*\} < D_1$$

and we have

$$\|W(t;\boldsymbol{v}(\cdot))\|_\omega \le D_1\delta(\boldsymbol{h}(t)). \tag{4.40}$$

Let $\widetilde{W}(D_1) = \{\boldsymbol{w} \in C\left(\boldsymbol{R}; E^\perp(\widehat{\boldsymbol{h}}) \cap X^\omega\right)\ ;\ \|\boldsymbol{w}(t)\|_\omega \le D_1\delta(\boldsymbol{h}(t))\}$. Then, (4.40) shows W is a map from $\widetilde{W}(D_1)$ into $\widetilde{W}(D_1)$.

Now, we shall show W is a contraction on $\widetilde{W}(D_1)$. We have from (4.38)

$$\begin{aligned}
&\|W(t;\boldsymbol{w}(\cdot)) - W(t;\boldsymbol{v}(\cdot))\|_\omega \\
&\le C\int_{-\infty}^t \|T(t,s)\{\widetilde{G}(\boldsymbol{h}(s), \boldsymbol{w}(s)) - \widetilde{G}(\boldsymbol{h}(s), \boldsymbol{v}(s))\}\|_\omega ds
\end{aligned}$$

$$
\begin{aligned}
&\le C\int_{-\infty}^{t}\max\{(t-s)^{-\omega},1\}e^{-\gamma_1(t-s)}\|\widetilde{G}(\boldsymbol{h}(s),\boldsymbol{w}(s))-\widetilde{G}(\boldsymbol{h}(s),\boldsymbol{v}(s))\|ds\\
&\le C\int_{-\infty}^{t}\max\{(t-s)^{-\omega},1\}e^{-\gamma_1(t-s)}\delta(\boldsymbol{h}(s))\|\boldsymbol{w}(s)-\boldsymbol{v}(s)\|_\omega ds\\
&\le C\int_{0}^{\infty}\max\{s^{-\omega},1\}e^{-\gamma_1' s}\{e^{-(\gamma_1-\gamma_1')s}\delta(\boldsymbol{h}(t-s))\}ds\cdot\sup_t\|\boldsymbol{w}(t)-\boldsymbol{v}(t)\|_\omega\\
&\le C\delta(\boldsymbol{h}(t))\sup_t\|\boldsymbol{w}(t)-\boldsymbol{v}(t)\|_\omega\\
&\le C\delta^*\sup_t\|\boldsymbol{w}(t)-\boldsymbol{v}(t)\|_\omega.
\end{aligned}
$$

This shows W is a contraction on $\widetilde{W}(D_1)$ if h^* is large enough, which completes the proof. ∎

Fix D_1 such that Lemma 4.6 holds.

Define

$$
J(\sigma)(\xi)=\boldsymbol{w}(0;\boldsymbol{h}(\cdot;\xi,\sigma))
$$

for $\xi\in\boldsymbol{R}^N$ and $\sigma\in\widetilde{W}(D_1,D_2)$. Then,

$$
\|J(\sigma)(\xi)\|_\omega\le D_1\delta(\boldsymbol{h}(0;\xi,\sigma))=D_1\delta(\xi) \tag{4.41}
$$

holds by the definition and Lemma 4.6.

We shall estimate $\|J(\sigma)(\xi_2)-J(\sigma)(\xi_1)\|_\omega$ for $\xi_2,\ \xi_1\in\boldsymbol{R}^N$ and $\sigma\in\widetilde{W}(D_1,D_2)$.

Lemma 4.7. *If* $\boldsymbol{h}_2,\boldsymbol{h}_1\in C^1(\boldsymbol{R};\boldsymbol{R}^N)$ *with* $|\frac{d}{dt}\boldsymbol{h}_j|\le C_3\delta(\boldsymbol{h}_j(t))$ *satisfy*

$$
|\boldsymbol{h}_2(t)-\boldsymbol{h}_1(t)|\le\zeta e^{\gamma_2\delta^*|t|}
$$

for positive constants ζ *and* γ_2*, then*

$$
\|\boldsymbol{w}(t;\boldsymbol{h}_2(\cdot))-\boldsymbol{w}(t;\boldsymbol{h}_1(\cdot))\|_\omega\le\delta(\boldsymbol{h}_1(t),\boldsymbol{h}_2(t))C\zeta e^{\gamma_2\delta^*|t|}
$$

holds.

Proof. Let $\boldsymbol{w}_j(t)=\boldsymbol{w}(t;\boldsymbol{h}_j(\cdot))$ $(j=1,2)$. Since $\boldsymbol{w}_j$ satisfy

$$
\boldsymbol{w}_j(t)=\int_{-\infty}^{t}T(t,s)\widetilde{G}(\boldsymbol{h}_j(s),\boldsymbol{w}_j(s))ds
$$

from (4.39), we have by using similar arguments to (4.39)

$$
\begin{aligned}
&\|\boldsymbol{w}_2(t)-\boldsymbol{w}_1(t)\|_\omega\\
&\le C\int_{-\infty}^{t}\|T(t,s)\{\widetilde{G}(\boldsymbol{h}_2(s),\boldsymbol{w}_2(s))-\widetilde{G}(\boldsymbol{h}_1(s),\boldsymbol{w}_1(s))\}\|_\omega ds\\
&\le C\int_{-\infty}^{t}\max\{(t-s)^{-\omega},1\}e^{-\gamma_1(t-s)}\|\widetilde{G}(\boldsymbol{h}_2(s),\boldsymbol{w}_2(s))-\widetilde{G}(\boldsymbol{h}_1(s),\boldsymbol{w}_1(s))\|ds\\
&\le C\int_{-\infty}^{t}\max\{(t-s)^{-\omega},1\}e^{-\gamma_1(t-s)}\delta(\boldsymbol{h}_2(s),\boldsymbol{h}_1(s))\{|\boldsymbol{h}_2(s)-\boldsymbol{h}_1(s)|+\|\boldsymbol{w}_2(s)-\boldsymbol{w}_1(s)\|_\omega\}ds\\
&\le C\delta(\boldsymbol{h}_2(t),\boldsymbol{h}_1(t))\int_{-\infty}^{t}\max\{(t-s)^{-\omega},1\}e^{-\gamma_1'(t-s)}\zeta e^{\gamma_2\delta^*|s|}ds\\
&\qquad+C\delta(\boldsymbol{h}_2(t),\boldsymbol{h}_1(t))\int_{-\infty}^{t}\max\{(t-s)^{-\omega},1\}e^{-\gamma_1'(t-s)}\|\boldsymbol{w}_2(s)-\boldsymbol{w}_1(s)\|_\omega ds\\
&\le C\delta(\boldsymbol{h}_2(t),\boldsymbol{h}_1(t))\int_{0}^{\infty}\max\{s^{-\omega},1\}e^{-\gamma_1' s}\zeta e^{\gamma_2\delta^*|t-s|}ds
\end{aligned}
$$

$$+C\delta(\boldsymbol{h}_2(t),\boldsymbol{h}_1(t))\int_{-\infty}^{t}\max\{(t-s)^{-\omega},1\}e^{-\gamma_1'(t-s)}e^{\gamma_2\delta^*|s|}ds\cdot\sup_t\{e^{-\gamma_2\delta^*|t|}\|\boldsymbol{w}_2(t)-\boldsymbol{w}_1(t)\|_\omega\}$$
$$\le C\delta(\boldsymbol{h}_2(t),\boldsymbol{h}_1(t))\zeta e^{\gamma_2\delta^*|t|}\int_0^\infty\max\{s^{-\omega},1\}e^{-(\gamma_1'-\gamma_2\delta^*)s}ds$$
$$+C\delta(\boldsymbol{h}_2(t),\boldsymbol{h}_1(t))\int_0^\infty\max\{s^{-\omega},1\}e^{-\gamma_1's}e^{\gamma_2\delta^*|t-s|}ds\cdot\sup_t\{e^{-\gamma_2\delta^*|t|}\|\boldsymbol{w}_2(t)-\boldsymbol{w}_1(t)\|_\omega\}$$
$$\le C\delta(\boldsymbol{h}_2(t),\boldsymbol{h}_1(t))e^{\gamma_2\delta^*|t|}\{\zeta+\sup_t(e^{-\gamma_2\delta^*|t|}\|\boldsymbol{w}_2(t)-\boldsymbol{w}_1(t)\|_\omega)\}$$

for a positive constant $0<\gamma_1'<\gamma_1$. This yields

$$\sup_t(e^{-\gamma_2\delta^*|t|}\|\boldsymbol{w}_2(t)-\boldsymbol{w}_1(t)\|_\omega\}\le C\delta(\boldsymbol{h}_2(t),\boldsymbol{h}_1(t))\{\zeta+\sup_t(e^{-\gamma_2\delta^*|t|}\|\boldsymbol{w}_2(t)-\boldsymbol{w}_1(t)\|_\omega)\},$$

which completes the proof. ∎

Let $\boldsymbol{h}_j(t)=\boldsymbol{h}(t;\xi_j,\sigma)$ $(j=1,2)$ for $\xi_j\in\boldsymbol{R}^N$ and $\sigma\in\widetilde{W}(D_1,D_2)$. Let $\boldsymbol{w}_j(t)=\sigma(\boldsymbol{h}_j(t))$.

Lemma 4.8. *$\boldsymbol{h}_j(t)$ $(j=1,2)$ defined above satisfy*

$$|\boldsymbol{h}_2(t)-\boldsymbol{h}_1(t)|\le e^{\delta^*C_4(1+2\delta^*D_2)|t|}|\xi_2-\xi_1|.$$

Proof. Let $\boldsymbol{h}_3(t)=\boldsymbol{h}_2(t)-\boldsymbol{h}_1(t)$. From (4.29), we have

$$\left|\frac{d}{dt}\boldsymbol{h}_3\right|\le\delta^*C_4\{|\boldsymbol{h}_3(t)|+\|\boldsymbol{w}_2(t)-\boldsymbol{w}_1(t)\|_\omega\}.$$

Since

$$\begin{aligned}\|\boldsymbol{w}_2(t)-\boldsymbol{w}_1(t)\|_\omega&=\|\sigma(\boldsymbol{h}_2(t))-\sigma(\boldsymbol{h}_1(t))\|_\omega\\&\le 2\delta^*D_2|\boldsymbol{h}_3(t)|\end{aligned}$$

holds, it follows that

$$\left|\frac{d}{dt}\boldsymbol{h}_3\right|\le\delta^*C_4(1+2\delta^*D_2)|\boldsymbol{h}_3|.$$

This gives the proof. ∎

From Lemmas 4.7 and 4.8,

$$\|\boldsymbol{w}(t;\boldsymbol{h}_2(\cdot))-\boldsymbol{w}(t;\boldsymbol{h}_1(\cdot))\|_\omega\le\delta(\boldsymbol{h}_2(t),\boldsymbol{h}_1(t))C|\xi_2-\xi_1|e^{\delta^*C_4(1+2\delta^*D_2)|t|}$$

holds. Therefore, we have

$$\begin{aligned}\|J(\sigma)(\xi_2)-J(\sigma)(\xi_1)\|_\omega&=\|\boldsymbol{w}(0;\boldsymbol{h}_2(\cdot))-\boldsymbol{w}(0;\boldsymbol{h}_1(\cdot))\|_\omega\\&\le\delta(\boldsymbol{h}_2(t),\boldsymbol{h}_1(t))C|\xi_2-\xi_1|\\&\le D_2\delta(\boldsymbol{h}_2(t),\boldsymbol{h}_1(t))|\xi_2-\xi_1|\end{aligned}\tag{4.42}$$

by taking D_2 appropriately large. (4.41) and (4.42) imply that

$$J(\sigma):\widetilde{W}(D_1,D_2)\to\widetilde{W}(D_1,D_2).\tag{4.43}$$

We shall show J is a contraction map on $\widetilde{W}(D_1,D_2)$. For given $\sigma_1,\ \sigma_2\in\widetilde{W}(D_1,D_2)$ and $\xi\in\boldsymbol{R}^N$, we let $\boldsymbol{h}_j(t)=\boldsymbol{h}(t;\xi,\sigma_j)$ and $\boldsymbol{w}_j(t)=\sigma_j(\boldsymbol{h}_j(t))$ $(j=1,2)$.

LEMMA 4.9.

$$|\boldsymbol{h}_2(t)-\boldsymbol{h}_1(t)| \le \frac{1}{1+2\delta^* D_2}|||\sigma_2-\sigma_1|||e^{\delta^* C_4(1+2\delta^* D_2)|t|},$$

holds, where $|||\cdot|||$ *denotes* $|||\sigma||| = \sup_{\xi\in\boldsymbol{R}^N}\|\sigma(\xi)\|_\omega$ *for* $\sigma \in \widetilde{W}(D_1, D_2)$.

Proof. Let $\boldsymbol{h}_3(t) = \boldsymbol{h}_2(t) - \boldsymbol{h}_1(t)$. Then, from (4.29), we have

$$
\begin{aligned}
|\frac{d}{dt}\boldsymbol{h}_3| &\le C_4\delta^*\{|\boldsymbol{h}_3| + \|\boldsymbol{w}_2(t)-\boldsymbol{w}_1(t)\|_\omega\} \\
&= C_4\delta^*\{|\boldsymbol{h}_3| + \|\sigma_2(\boldsymbol{h}_2(t))-\sigma_1(\boldsymbol{h}_1(t))\|_\omega\}.
\end{aligned} \tag{4.44}
$$

Here,

$$
\begin{aligned}
\|\sigma_2(\boldsymbol{h}_2(t))-\sigma_1(\boldsymbol{h}_1(t))\|_\omega &\le \|\sigma_1(\boldsymbol{h}_2(t))-\sigma_1(\boldsymbol{h}_1(t))\|_\omega + \|\sigma_2(\boldsymbol{h}_2(t))-\sigma_1(\boldsymbol{h}_2(t))\|_\omega \\
&\le D_2\delta(\boldsymbol{h}_2(t),\boldsymbol{h}_1(t))|\boldsymbol{h}_2(t)-\boldsymbol{h}_1(t)| + |||\sigma_2-\sigma_1||| \\
&\le 2D_2\delta^*|\boldsymbol{h}_2(t)-\boldsymbol{h}_1(t)| + |||\sigma_2-\sigma_1|||
\end{aligned}
$$

holds. Substituting this to (4.44), we have

$$|\frac{d}{dt}\boldsymbol{h}_3| \le C_4\delta^*(1+2\delta^* D_2)|\boldsymbol{h}_3| + C_4\delta^*|||\sigma_2-\sigma_1|||.$$

This yields

$$
\begin{aligned}
|\boldsymbol{h}_3(t)| &\le \int_0^{|t|} e^{C_4\delta^*(1+2\delta^* D_2)(|t|-s)}ds\cdot C_4\delta^*|||\sigma_2-\sigma_1||| \\
&= \frac{1}{1+2\delta^* D_2}\left(e^{C_4\delta^*(1+2\delta^* D_2)|t|}-1\right)|||\sigma_2-\sigma_1||| \\
&\le \frac{1}{1+2\delta^* D_2}|||\sigma_2-\sigma_1|||e^{C_4\delta^*(1+2\delta^* D_2)|t|}.
\end{aligned}
$$

∎

Lemmas 4.7 and 4.9 imply

$$\|\boldsymbol{w}(t;\boldsymbol{h}_2(\cdot))-\boldsymbol{w}(t;\boldsymbol{h}_1(\cdot))\|_\omega \le \frac{2\delta^* C}{1+2\delta^* D_2}|||\sigma_2-\sigma_1|||e^{C_4\delta^*(1+2\delta^* D_2)|t|}.$$

This directly shows

$$
\begin{aligned}
\|J(\sigma_2)(\xi)-J(\sigma_1)(\xi)\|_\omega &= \|\boldsymbol{w}(0;\boldsymbol{h}_2(\cdot))-\boldsymbol{w}(0;\boldsymbol{h}_1(\cdot))\|_\omega \\
&\le \frac{2\delta^* C}{1+2\delta^* D_2}|||\sigma_2-\sigma_1||| \\
&\le C\delta^*|||\sigma_2-\sigma_1|||.
\end{aligned} \tag{4.45}
$$

Hence, J is a contraction and there uniquely exists $\widehat{\sigma} \in \widetilde{W}(D_1, D_2)$ satisfying $J(\widehat{\sigma}) = \widehat{\sigma}$.

Let $\widehat{\mathcal{M}}(\widehat{\boldsymbol{h}}, \rho_1) = \{\Xi(l)[P(z;\boldsymbol{h}) + \Pi(\boldsymbol{h})\widehat{\sigma}(\boldsymbol{h})]\ ;\ l \in \boldsymbol{R}^1,\ \boldsymbol{h} \in H(\widehat{\boldsymbol{h}}, \rho_1)\}$. Then, from the construction of $\widehat{\sigma}$, we can show easily that $\widehat{\mathcal{M}}(\widehat{\boldsymbol{h}}, \rho_1)$ is positively invariant with respect to the flow of (1.6) as long as $\boldsymbol{h}(t) \in H(\widehat{\boldsymbol{h}}, \rho_1)$, where $\boldsymbol{h}(t)$ is a solution of (4.34) with $\sigma = \widehat{\sigma}$. Smoothness and an exponential attractivity of $\widehat{\mathcal{M}}(\widehat{\boldsymbol{h}}, \rho_1)$ together with the existence of aymptotic phase are also shown in quite a similar manner to Section 9 of [19]. Here, we specially note that the attractivity is determined only by the estimate of semigroup $e^{-A(\boldsymbol{h})t}$. This implies that the set $\widehat{\mathcal{M}}(\widehat{\boldsymbol{h}}, \rho_1)$ has an attractivity uniformly for $\widehat{\boldsymbol{h}}$ with sufficiently large $\min\widehat{\boldsymbol{h}} > h^*$. That is, we have the following result now.

THEOREM 4.1. *There exist positive constants h^*, ρ_1, γ_3, η_1 and M_3 such that for any $\widehat{\boldsymbol{h}}$ with $\min \widehat{\boldsymbol{h}} \geq h^*$ and any $\boldsymbol{u}(0,\cdot)$ with $dist\{\boldsymbol{u}(0,\cdot),\ \widehat{\mathcal{M}}(\widehat{\boldsymbol{h}},\rho_1)\} < \eta_1$, there exist functions $l(t)$ and $\boldsymbol{h}(t)$ which are the solutions of (4.17), (4.18) with $\boldsymbol{w} = \widehat{\sigma}(\boldsymbol{h})$ such that*

$$\|\boldsymbol{u}(t,\cdot) - \Xi(l(t))\{P(z;\boldsymbol{h}(t)) + \Pi(\boldsymbol{h}(t))\widehat{\sigma}(\boldsymbol{h}(t))\}\|_\omega \leq M_3 e^{-\gamma_3 t} dist\{\boldsymbol{u}(0,\cdot),\ \widehat{\mathcal{M}}(\widehat{\boldsymbol{h}},\rho_1)\}$$

holds as long as $\boldsymbol{h}(t) \in H(\widehat{\boldsymbol{h}},\rho_1)$, where $\boldsymbol{u}(t,z)$ is a solution of (1.6).

By using this theorem, we shall show Theorems 2.1 and 2.2.

4.2. Proof of Thorem 2.1. Let $\rho_3 = \frac{1}{2}\rho_1$. For any $\boldsymbol{k} = (k_1, k_2, \cdots, k_N) \in \boldsymbol{Z}_+^N$, define $\widehat{h}_j(\boldsymbol{k}) = h^* + k_j\rho_3$ and $\widehat{\boldsymbol{h}}(\boldsymbol{k}) = (\widehat{h}_1(\boldsymbol{k}), \widehat{h}_2(\boldsymbol{k}), \cdots, \widehat{h}_N(\boldsymbol{k}))$, where $\boldsymbol{Z}_+ = \{0, 1, \cdots\}$. Then, $(h^*,\infty)^N = \cup_{\boldsymbol{k}\in\boldsymbol{Z}_+^N} H(\widehat{\boldsymbol{h}}(\boldsymbol{k}),\rho_1)$ holds. Theorem 4.1 shows that there exist, say $\widehat{\sigma} = \widehat{\sigma}(\cdot;\boldsymbol{k})$, and $\widehat{\mathcal{M}}(\widehat{\boldsymbol{h}}(\boldsymbol{k}),\rho_1)$ for any $\boldsymbol{k} \in \boldsymbol{Z}_+^N$ such that the solution $\boldsymbol{u}(t,\cdot)$ of (1.6) satisfies

$$\|\boldsymbol{u}(t,\cdot) - \Xi(l(t))\{P(\cdot;\boldsymbol{h}(t)) + \Pi(\boldsymbol{h}(t))\widehat{\sigma}(\boldsymbol{h}(t);\boldsymbol{k})\}\|_\omega \leq M_3 e^{-\gamma_3 t} dist\{\boldsymbol{u}(0,\cdot),\ \widehat{\mathcal{M}}(\widehat{\boldsymbol{h}},\rho_1)\},$$

and therefore,

$$\begin{aligned}
&\|\boldsymbol{u}(t,\cdot) - \Xi(l(t))P(\cdot;\boldsymbol{h}(t))\|_\omega \\
&\leq M_3 e^{-\gamma_3 t} dist\{\boldsymbol{u}(0,\cdot),\ \widehat{\mathcal{M}}(\widehat{\boldsymbol{h}},\rho_1)\} + \|\Xi(l(t))\Pi(\boldsymbol{h}(t))\widehat{\sigma}(\boldsymbol{h}(t);\boldsymbol{k})\|_\omega \\
&\leq M_3 e^{-\gamma_3 t} dist\{\boldsymbol{u}(0,\cdot),\ \widehat{\mathcal{M}}(\widehat{\boldsymbol{h}},\rho_1)\} + C\delta(\boldsymbol{h}(t)) \\
&\leq C\delta(\boldsymbol{h}(t))
\end{aligned}$$

holds for sufficiently large $t > 0$ because $\widehat{\sigma}(\cdot;\boldsymbol{k}) \in \widetilde{W}(D_1, D_2)$. Since we may assume $C\delta(\boldsymbol{h}(t)) \leq C\delta^* < \eta_1$, the solution $\boldsymbol{u}(t,\cdot)$ is in an attractive region of $\widehat{\mathcal{M}}(\widehat{\boldsymbol{h}}(\boldsymbol{k}),\rho_1)$ for a certain $\boldsymbol{k}$. That is, there exists $\boldsymbol{k} \in H(\widehat{\boldsymbol{h}}(\boldsymbol{k}),\rho_1)$ for any $\boldsymbol{u}(t,\cdot)$ with $\boldsymbol{h}(t) \in (h^*,\infty)^N$ such that $\boldsymbol{u}(t,\cdot)$ stays close to $\widehat{\mathcal{M}}(\widehat{\boldsymbol{h}}(\boldsymbol{k}),\rho_1)$. This completes the proof.

4.3. Proof of Thorem 2.2. Let $\overline{\boldsymbol{h}}$ be the equilibrium stated in the theorem. Since $\min \overline{\boldsymbol{h}} > h^*$ is satisfied, there exist a $\overline{\boldsymbol{k}} \in (h^*,\infty)^N$ such that $\overline{\boldsymbol{h}} \in H(\widehat{\boldsymbol{h}}(\overline{\boldsymbol{k}}),\rho_1)$ and there exist a function $\overline{\sigma}(\boldsymbol{h}) = \widehat{\sigma}(\boldsymbol{h};\overline{\boldsymbol{k}})$ which gives an attractive invariant set of (4.32) with $\boldsymbol{w} = \overline{\sigma}(\boldsymbol{h})$ by Theorem 4.1. As

$$\widetilde{\boldsymbol{H}}(\boldsymbol{h},\overline{\sigma}(\boldsymbol{h})) = \boldsymbol{H}(\boldsymbol{h}) + O(\delta^2(\boldsymbol{h}))$$

holds from (4.18), it is easily shown by the implicit function theorem that $\widetilde{\boldsymbol{H}}(\boldsymbol{h},\overline{\sigma}(\boldsymbol{h}))$ has a stable equilibrium $\boldsymbol{h}^\dagger$ satisfying

$$\boldsymbol{h}^\dagger = \overline{\boldsymbol{h}} + O(\delta(\overline{\boldsymbol{h}})).$$

Defining $l^\dagger = \widetilde{H}_0(\boldsymbol{h}^\dagger,\overline{\sigma}(\boldsymbol{h}^\dagger))$ and $\overline{P}(\cdot) = P(\cdot,\boldsymbol{h}^\dagger) + \Pi(\boldsymbol{h}^\dagger)\overline{\sigma}(\boldsymbol{h}^\dagger)$, we see

$$\Xi(l^\dagger t)\overline{P}(\cdot) = \overline{P}(z - l^\dagger t)$$

gives the stable traveling wave pulse solution stated in the statement.

Unstable traveling wave pulse solution is constructed in quite a similar manner.

4.4. Proofs of theorems when D has zero elements. In this subsection, we use same notations and symbols as those in previous subsections as long as they are not specially noticed.

When D includes zero in the elements, the operator $L(\boldsymbol{h})$ is not sectorial and the fractional powered space X^ω imbedded into $BU^1(\boldsymbol{R})$ is not defined, and hence, the transformation of (4.10) and (4.11) is not applicable. We reconstruct a new map instead of $\Pi(\boldsymbol{h})$.

Let the base space $X = BU^0(\boldsymbol{R}^1)$ with sup-norm and let an operator $K(\boldsymbol{h})$ be

$$K(\boldsymbol{h})\boldsymbol{v} = \sum_{j=0}^{N} \langle\ \boldsymbol{v}, \frac{\partial}{\partial z}\boldsymbol{\phi}_j^*(\boldsymbol{h})\ \rangle_{L^2}\, \boldsymbol{\phi}_j(\boldsymbol{h})$$

for $\boldsymbol{v} \in X$. Fix $\hat{l} \in \boldsymbol{R}^1$ arbitrarily and define a map $\Lambda(l, \boldsymbol{h})$ by $\Lambda(l, \boldsymbol{h})\boldsymbol{v}_0 = \boldsymbol{v}(l)$, where $\boldsymbol{v}(r)$ is a solution of

$$\begin{cases} \dfrac{d\boldsymbol{v}}{dr} &= \Xi(r)K(\boldsymbol{h})\Xi(-r)\boldsymbol{v}, \\ \boldsymbol{v}(\hat{l}) &= \boldsymbol{v}_0 \in X. \end{cases}$$

Let $\widehat{\Pi}(l, \boldsymbol{h}) = \Lambda(l, \boldsymbol{h})\Pi(\boldsymbol{h})$ and $E^{\perp}(l, \boldsymbol{h}) = \Xi(l)E^{\perp}(\boldsymbol{h})$. Then, it is easiy shown that the map $\widehat{\Pi}(l, \boldsymbol{h})$ is homeomorphic from $E^{\perp}(\hat{l}, \widehat{\boldsymbol{h}})$ to $E^{\perp}(l, \boldsymbol{h})$ by (4.3) and (4.4). Moreover, it is also shown by the construction that $\widehat{\Pi}(l, \boldsymbol{h})$ is a bounded operator on X up to their first derivatives with respect to l and $\boldsymbol{h}$. Hence, the transformation

$$\boldsymbol{v} = \Xi(l)P(z; \boldsymbol{h}) + \widehat{\Pi}(l, \boldsymbol{h})\boldsymbol{w}$$

for $(l, \boldsymbol{h}) \in H(\hat{l}, \widehat{\boldsymbol{h}}, \rho_1)$ and $\boldsymbol{w} \in E^{\perp}(\hat{l}, \widehat{\boldsymbol{h}})$, where $H(\hat{l}, \widehat{\boldsymbol{h}}, \rho_1) = \{(l, \boldsymbol{h}) \in \boldsymbol{R}^{N+1};\ \hat{l} < l < \hat{l} + \rho_1,\ \boldsymbol{h} \in H(\widehat{\boldsymbol{h}}, \rho_1)\}$ yields the equations of $(l, \boldsymbol{h}, \boldsymbol{w})$ as (4.17), (4.18) and (4.20)

$$\dot{l} = \widetilde{H}_0(l, \boldsymbol{h}, \boldsymbol{w}) = -H_0(\boldsymbol{h}) + O(\delta^2), \tag{4.46}$$

$$\begin{aligned} \dot{h}_j &= \widetilde{H}_j(l, \boldsymbol{h}, \boldsymbol{w}) \\ &= H_{j-1}(\boldsymbol{h}) - H_j(\boldsymbol{h}) + O(\delta^2), \end{aligned} \tag{4.47}$$

$$\boldsymbol{w}_t = A(l, \boldsymbol{h})\boldsymbol{w} + \widetilde{G}(l, \boldsymbol{h}, \boldsymbol{w}) \tag{4.48}$$

for $\boldsymbol{w} \in E^{\perp}(\hat{l}, \widehat{\boldsymbol{h}})$ with $\|\boldsymbol{w}\|_\infty = O(\delta)$, where $A(l, \boldsymbol{h}) = \widehat{\Pi}^{-1}(l, \boldsymbol{h})\Xi(l)L(\boldsymbol{h})\widehat{\Pi}(l, \boldsymbol{h})$ and so on. Then, quite a similar way to the previous subsections, we can show the existence of $\widehat{\sigma}(l, \boldsymbol{h}) \in E^{\perp}(\hat{l}, \widehat{\boldsymbol{h}})$ and a positively locally invariant attractive manifold given by

$$\widehat{M}(\hat{l}, \widehat{\boldsymbol{h}}, \rho_1) = \{\Xi(l)P(z; \boldsymbol{h}) + \widehat{\Pi}(l, \boldsymbol{h})\widehat{\sigma}(l, \boldsymbol{h});\ (l, \boldsymbol{h}) \in H(\hat{l}, \widehat{\boldsymbol{h}}, \rho_1)\}$$

such that $\|\widehat{\sigma}(l, \boldsymbol{h})\|_\infty \le C\delta(\boldsymbol{h})$ and

$$\|\boldsymbol{u}(t, \cdot) - \{\Xi(l)P(z; \boldsymbol{h}) + \widehat{\Pi}(l, \boldsymbol{h})\widehat{\sigma}(l, \boldsymbol{h})\}\|_\infty \le Ce^{-\gamma' t} dist\{\boldsymbol{u}(0, \cdot), \widehat{M}(\hat{l}, \widehat{\boldsymbol{h}}, \rho_1)\}$$

holds as long as $(l, \boldsymbol{h}) \in H(\hat{l}, \widehat{\boldsymbol{h}}, \rho_1)$ if $\boldsymbol{u}(0, \cdot)$ is sufficiently close to $\widehat{M}(\hat{l}, \widehat{\boldsymbol{h}}, \rho_1)$. In the proof of these results, we used the same estimate for the operator $A(l, \boldsymbol{h})$ as in Lemma 4.5 with $\omega = 0$ though A generates only the C^0 senigroup. In fact, the estimate in Lemma 4.5 with $\omega = 0$ is obtained only from the result in Lemma 4.4 and the exponential decay of e^{tA} with respect to $t > 0$, which has been extensively studied in related works on the stabiliy of the 1-pulse solution for the FitzHugh-Nagumo systems (e.g. [5] $\sim$ [8], [17],[30]).

The rest of the proof is the same as subsection 4.2.

4.5. Proof of Thorem 2.3. From the assumptions (2.5), (2.6) on $P(z)$, constants α, β and vectors $\boldsymbol{a}^{\pm}$ satisfy

$$\alpha^2 D\boldsymbol{a}^+ + \theta\alpha\boldsymbol{a}^+ + F'(0)\boldsymbol{a}^+ = 0, \tag{4.49}$$

$$\beta^2 D\boldsymbol{a}^- - \theta\beta\boldsymbol{a}^- + F'(0)\boldsymbol{a}^- = 0. \tag{4.50}$$

Hence, it is easily seen that there exist non zero eigenvectors $\boldsymbol{b}^{\pm}$ satisfying

$$\alpha^2 D\boldsymbol{b}^- + \theta\alpha\boldsymbol{b}^- + {}^tF'(0)\boldsymbol{b}^- = 0, \tag{4.51}$$

$$\beta^2 D\boldsymbol{b}^+ - \theta\beta\boldsymbol{b}^+ + {}^tF'(0)\boldsymbol{b}^+ = 0. \tag{4.52}$$

(4.51) and (4.52) give the asymptotic form of $\boldsymbol{\phi}^*$ as $x \to \mp\infty$, respectively, which shows (2.7) and (2.8) hold.

We fix j^* $(1 \le j^* \le N-1)$ arbitrarily and show (2.9) for $j = j^*$. For $j = 0$ and N, we can prove (2.10), (2.11) in quite a similar way and omit the proof.

Let $h_{min} = \min \boldsymbol{h}$ and

$$\mu = \frac{\beta}{2(\alpha+\beta)} h_{min}. \tag{4.53}$$

Defining $I_1 = (-\infty,\ -h_{j^*} + \mu] \cup [h_{j^*+1} - \mu,\ \infty)$ and $I_2 = (-h_{j^*} + \mu,\ h_{j^*+1} - \mu)$, we consider

$$\begin{aligned} H_{j^*}(\boldsymbol{h}) &= \int_{-\infty}^{\infty} \langle\ \mathcal{L}(P(z + z_{j^*}; \boldsymbol{h})), \boldsymbol{\phi}^*(z)\ \rangle\, dz \\ &= \int_{I_1 \cup I_2} \langle\ \mathcal{L}(P(z + z_{j^*}; \boldsymbol{h})), \boldsymbol{\phi}^*(z)\ \rangle\, dz. \end{aligned}$$

Lemma 4.10.

$$\left| \int_{I_1} \langle\ \mathcal{L}(P(z + z_{j^*}; \boldsymbol{h})), \boldsymbol{\phi}^*(z)\ \rangle\, dz \right| \le C \left(e^{-\alpha h_{j^*}} + e^{-\beta h_{j^*+1}} \right) e^{-\gamma_4 h_{min}}$$

for a constant $\gamma_4 > 0$.

Proof.

Let $I_1 = I_1^- \cup I_1^+$, where $I_1^- = (-\infty,\ -h_{j^*} + \mu]$ and $I_1^+ = [h_{j^*+1} - \mu,\ \infty)$. We divide I_1^- moreover into $I_1^- = I_1^{-,L} \cup I_1^{-,R}$, where $I_1^{-,L} = (-\infty,\ -h_{j^*}]$ and $I_1^{-,R} = (-h_{j^*},\ -h_{j^*} + \mu]$. On $I_1^{-,L}$, $\boldsymbol{\phi}^*(z)$ has the estimate (2.8) and we have

$$\begin{aligned} &\left| \int_{I_1^{-,L}} \langle\ \mathcal{L}(P(z + z_{j^*}; \boldsymbol{h})), \boldsymbol{\phi}^*(z)\ \rangle\, dz \right| \\ &\le C\delta(\boldsymbol{h}) \int_{I_1^{-,L}} |\boldsymbol{\phi}^*(z)|\, dz \\ &\le C\delta(\boldsymbol{h}) e^{-\alpha h_{j^*}} \left(1 + O(e^{-\gamma h_{j^*}})\right) \\ &\le C\delta(\boldsymbol{h}) e^{-\alpha h_{j^*}}. \end{aligned} \tag{4.54}$$

On the other hand, on $I_1^{-,R}$,

$$|P(z + z_{j^*}; \boldsymbol{h}) - P(z + h_{j^*})| \le C \left(e^{-\alpha(z + h_{j^*} + h_{j^*-1})} + e^{\beta z} \right)$$

and for $j \ne j^* - 1$,

$$|P(z + z_{j^*} - z_j)| \le C \left(e^{-\alpha(z + h_{j^*} + h_{j^*-1})} + e^{\beta z} \right)$$

hold. Hence, we have for $z \in I_1^{-,R}$,

$$\begin{aligned} \mathcal{L}(P(z + z_{j^*}; \boldsymbol{h})) &= \mathcal{L}(P(z + z_{j^*}; \boldsymbol{h})) - \sum_{j=0}^{N} \mathcal{L}(P(z + z_{j^*} - z_j)) \\ &= F(P(z + z_{j^*}; \boldsymbol{h})) - \sum_{j=0}^{N} F(P(z + z_{j^*} - z_j)) \\ &= F(P(z + h_{j^*})) + F'(P(z + h_{j^*})) \sum_{j \ne j^*-1}^{N} P(z + z_{j^*} - z_j) \\ &\qquad - F(P(z + h_{j^*})) - \sum_{j \ne j^*-1}^{N} F'(\mathbf{0}) P(z + z_{j^*} - z_j) \\ &\qquad + O\left(e^{-2\alpha(z + h_{j^*} + h_{j^*-1})} + e^{2\beta z} \right) \end{aligned}$$

$$= \{F'(P(z+h_{j^*})) - F'(\mathbf{0})\} \sum_{j \neq j^*-1}^{N} P(z+z_{j^*}-z_j)$$
$$+O\left(e^{-2\alpha(z+h_{j^*}+h_{j^*-1})} + e^{2\beta z}\right)$$
$$= O\left(e^{-\alpha(z+h_{j^*}+h_{j^*-1})} + e^{\beta z}\right).$$

Thus, it follows that

$$\left| \int_{I_1^{-,R}} \langle\, \mathcal{L}(P(z+z_{j^*};\boldsymbol{h})), \boldsymbol{\phi}^*(z) \,\rangle \, dz \right| \tag{4.55}$$
$$\leq C \int_{I_1^{-,R}} \left(e^{-\alpha(z+h_{j^*}+h_{j^*-1})} + e^{\beta z}\right) \cdot |\boldsymbol{\phi}^*(z)| \, dz$$
$$\leq C \int_{I_1^{-,R}} \left(e^{-\alpha(z+h_{j^*}+h_{j^*-1})} + e^{\beta z}\right) \cdot e^{\alpha z} \left(1 + O(e^{\gamma z})\right) dz$$
$$\leq C \left(\mu e^{-\alpha(h_{j^*}+h_{j^*-1})} + e^{-(\alpha+\beta)(h_{j^*}-\mu)}\right)$$
$$= C e^{-\alpha h_{j^*}} \left(\mu e^{-\alpha h_{j^*-1}} + e^{-\beta h_{j^*}+(\alpha+\beta)\mu}\right)$$
$$\leq C e^{-\alpha h_{j^*}} \left\{ \frac{\beta}{\alpha+\beta} h_{min} e^{-\alpha h_{j^*-1}} + e^{-\frac{1}{2}\beta h_{min}} \right\}$$
$$\leq C e^{-\alpha h_{j^*}} e^{-\gamma_5 h_{min}}$$

for a constant $\gamma_5 > 0$ because $h_{min} \leq h_{j^*-1}$ and (4.53). Since $\delta(\boldsymbol{h}) \leq C e^{-\gamma^* h_{min}}$ holds for $\gamma^* = \min\{\alpha, \beta\}$, (4.54) and (4.55) yield

$$\left| \int_{I_1^-} \langle\, \mathcal{L}(P(z+z_{j^*};\boldsymbol{h})), \boldsymbol{\phi}^*(z) \,\rangle \, dz \right| \leq C e^{-\alpha h_{j^*}} e^{-\gamma_6 h_{min}} \tag{4.56}$$

for a constant $\gamma_6 > 0$.

Similarly, it is shown that

$$\left| \int_{I_1^+} \langle\, \mathcal{L}(P(z+z_{j^*};\boldsymbol{h})), \boldsymbol{\phi}^*(z) \,\rangle \, dz \right| \leq C e^{-\beta h_{j^*}} e^{-\gamma_7 h_{min}} \tag{4.57}$$

for a constant $\gamma_7 > 0$.

(4.56) and (4.57) complete the proof. ∎

LEMMA 4.11.

$$\left| \int_{I_2} \langle\, \mathcal{L}(P(z+z_{j^*};\boldsymbol{h})), \boldsymbol{\phi}^*(z) \,\rangle \, dz \right| = \left(M_\alpha e^{-\alpha h_{j^*}} + M_\beta e^{-\beta h_{j^*+1}}\right) \left\{1 + O\left(e^{-\gamma_8 h_{min}}\right)\right\}$$

for a constant $\gamma_8 > 0$.

Proof.

From assumptions (2.5) and (2.6),

$$P(z+z_{j^*};\boldsymbol{h}) = P(z) + \sum_{j=0}^{j^*-1} e^{-\alpha(z+z_{j^*}-z_j)} \boldsymbol{a}^+ + \sum_{j=j^*+1}^{N} e^{\beta(z-z_{j^*}+z_j)} \boldsymbol{a}^- \tag{4.58}$$
$$+O(e^{-(\alpha+\gamma)(z+h_{j^*})} + e^{(\beta+\gamma)(z-h_{j^*+1})})$$

holds on I_2. We let the right hand side of (4.58) be $P(z)+P^L(z)+P^R(z)+g(z)$. Then, we have

$$\begin{aligned}
\mathcal{L}(P(z+z_{j^*};\boldsymbol{h})) &= \mathcal{L}(P(z+z_{j^*};\boldsymbol{h})) - \sum_{j=0}^{N}\mathcal{L}(P(z+z_{j^*}-z_j)) \\
&= F(P(z+z_{j^*};\boldsymbol{h})) - \sum_{j=0}^{N} F(P(z+z_{j^*}-z_j)) \\
&= F(P(z)) + F'(P(z))(P^L(z)+P^R(z)+g(z)) \\
&\qquad -\{F(P(z))+F'(\mathbf{0})(P^L(z)+P^R(z)+g(z))\} \\
&\qquad\quad +O(e^{-2\alpha(z+h_{j^*})}+e^{2\beta(z-h_{j^*+1})}) \\
&= \{F'(P(z))-F'(\mathbf{0})\}(P^L(z)+P^R(z))+g(z).
\end{aligned} \tag{4.59}$$

Here, we assumed $\gamma<\gamma^*$ and wrote $O(g(z))$ by $g(z)$ again.

It is easily seen that

$$\left|\int_{I_2}\langle\, g(z),\boldsymbol{\phi}^*(z)\,\rangle\, dz\right| \le C\left(e^{-\alpha h_{j^*}}+e^{-\beta h_{j^*+1}}\right)e^{-\gamma_9 h_{min}} \tag{4.60}$$

for a constant $\gamma_9>0$.

Now, we shall calculate

$$\int_{I_2}\langle\, \{F'(P(z))-F'(\mathbf{0})\}(P^L(z)+P^R(z)),\boldsymbol{\phi}^*(z)\,\rangle\, dz. \tag{4.61}$$

Substituting the definitions of $P^L(z)$ and $P^R(z)$ into (4.61), we have

$$\begin{aligned}
&\int_{I_2}\langle\, \{F'(P(z))-F'(\mathbf{0})\}(P^L(z)+P^R(z)),\boldsymbol{\phi}^*(z)\,\rangle\, dz \\
&= M'_\alpha\sum_{j=0}^{j^*-1} e^{-\alpha(z_{j^*}-z_j)} + M'_\beta\sum_{j=j^*+1}^{N} e^{\beta(-z_{j^*}+z_j)} \\
&= M'_\alpha e^{-\alpha h_{j^*}}\left(1+O(e^{-\alpha h_{min}})\right) + M'_\beta e^{-\beta h_{j^*+1}}\left(1+O(e^{-\beta h_{min}})\right) \\
&= \left(M'_\alpha e^{-\alpha h_{j^*}}+M'_\beta e^{-\beta h_{j^*+1}}\right)\left\{1+O(e^{-\gamma^* h_{min}})\right\},
\end{aligned} \tag{4.62}$$

where

$$\begin{aligned}
M'_\alpha &= \int_{I_2} e^{-\alpha z}\,\langle\, \{F'(P(z))-F'(\mathbf{0})\}\boldsymbol{a}^+,\boldsymbol{\phi}^*(z)\,\rangle\, dz, \\
M'_\beta &= \int_{I_2} e^{\beta z}\,\langle\, \{F'(P(z))-F'(\mathbf{0})\}\boldsymbol{a}^-,\boldsymbol{\phi}^*(z)\,\rangle\, dz.
\end{aligned}$$

Proposition 4.4.

$$\left|\int_{I_1} e^{-\alpha z}\,\langle\, \{F'(P(z))-F'(\mathbf{0})\}\boldsymbol{a}^+,\boldsymbol{\phi}^*(z)\,\rangle\, dz\right| \le Ce^{-\gamma_{10}h_{min}}, \tag{4.63}$$

$$\left|\int_{I_1} e^{\beta z}\,\langle\, \{F'(P(z))-F'(\mathbf{0})\}\boldsymbol{a}^-,\boldsymbol{\phi}^*(z)\,\rangle\, dz\right| \le Ce^{-\gamma_{10}h_{min}} \tag{4.64}$$

hold for a constant $\gamma_{10}>0$.

Proof.

We shall prove (4.63). On I_1^-, $P(z)$ and $\boldsymbol{\phi}^*(z)$ have estimates (2.6) and (2.8), respectively. Hence,

$$\left|\langle\, \{F'(P(z))-F'(\mathbf{0})\}\boldsymbol{a}^+,\boldsymbol{\phi}^*(z)\,\rangle\right| \le Ce^{\beta z}\cdot e^{\alpha z}$$

holds and therefore, we have

$$
\begin{aligned}
&\left| \int_{I_1} e^{\beta z} \langle \{F'(P(z)) - F'(\mathbf{0})\}\boldsymbol{a}^-, \boldsymbol{\phi}^*(z) \rangle \, dz \right| \\
&\le C \left| \int_{-\infty}^{-h_{j^*}+\mu} e^{-\alpha z} \cdot e^{\beta z} \cdot e^{\alpha z} dz \right| \\
&\le C e^{\beta(-h_{j^*}+\mu)} \\
&\le C e^{-\gamma_{10} h_{min}}.
\end{aligned}
$$

$|\langle \{F'(P(z)) - F'(\mathbf{0})\}\boldsymbol{a}^+, \boldsymbol{\phi}^*(z) \rangle|$ is estimated similarly on I_1^+ and (4.63) is proved. (4.64) is shown in quite a same manner. ∎

Propositon 4.4 implies that

$$
M'_\alpha = \int_{-\infty}^{\infty} e^{-\alpha z} \langle \{F'(P(z)) - F'(\mathbf{0})\}\boldsymbol{a}^+, \boldsymbol{\phi}^*(z) \rangle \, dz + O\left(e^{-\gamma_{10} h_{min}}\right), \tag{4.65}
$$

$$
M'_\beta = \int_{-\infty}^{\infty} e^{\beta z} \langle \{F'(P(z)) - F'(\mathbf{0})\}\boldsymbol{a}^-, \boldsymbol{\phi}^*(z) \rangle \, dz + O\left(e^{-\gamma_{10} h_{min}}\right). \tag{4.66}
$$

PROPOSITION 4.5.

$$
M_\alpha = \int_{-\infty}^{\infty} e^{-\alpha z} \langle \{F'(P(z)) - F'(\mathbf{0})\}\boldsymbol{a}^+, \boldsymbol{\phi}^*(z) \rangle \, dz, \tag{4.67}
$$

$$
M_\beta = \int_{-\infty}^{\infty} e^{\beta z} \langle \{F'(P(z)) - F'(\mathbf{0})\}\boldsymbol{a}^-, \boldsymbol{\phi}^*(z) \rangle \, dz \tag{4.68}
$$

hold.

Proof.

First, we will consider (4.67). Since $\boldsymbol{\phi}^*(z)$ and $\boldsymbol{a}^+$ satisfy $L^*(\boldsymbol{h})\boldsymbol{\phi}^* = 0$ and (4.49), we have

$$
\begin{aligned}
\langle F'(P(z))\boldsymbol{a}^+, \boldsymbol{\phi}^*(z) \rangle &= \langle \boldsymbol{a}^+, {}^tF'(P(z))\boldsymbol{\phi}^*(z) \rangle \\
&= -\langle \boldsymbol{a}^+, D\boldsymbol{\phi}^*_{zz} + \theta\boldsymbol{\phi}^*_z \rangle, \\
\langle F'(\mathbf{0})\boldsymbol{a}^+, \boldsymbol{\phi}^*(z) \rangle &= -\langle \alpha^2 D\boldsymbol{a}^+ + \theta\alpha\boldsymbol{a}^+, \boldsymbol{\phi}^*(z) \rangle \\
&= -\langle \boldsymbol{a}^+, \alpha^2 D\boldsymbol{\phi}^* + \theta\alpha\boldsymbol{\phi}^* \rangle.
\end{aligned}
$$

Hence, M_α is

$$
M_\alpha = \int_{-\infty}^{\infty} e^{-\alpha z} \langle \boldsymbol{a}^+, \alpha^2 D\boldsymbol{\phi}^* + \theta\alpha\boldsymbol{\phi}^* - D\boldsymbol{\phi}^*_{zz} - \theta\boldsymbol{\phi}^*_z \rangle \, dz. \tag{4.69}
$$

We calculate each term of the right hand side of (4.69). Note that (2.8) and

$$
\boldsymbol{\phi}^*_z(z) \to \alpha e^{\alpha z}\left(\boldsymbol{b}^- + O(e^{\gamma z})\right)
$$

hold as $z \to -\infty$. Then by integration by parts, we see

$$
\begin{aligned}
&\int_{-\infty}^{\infty} e^{-\alpha z} \langle \boldsymbol{a}^+, D\boldsymbol{\phi}^*_{zz} \rangle \, dz \\
&= \left[e^{-\alpha z} \langle \boldsymbol{a}^+, D\boldsymbol{\phi}^*_z \rangle\right]_{-\infty}^{\infty} + \alpha \int_{-\infty}^{\infty} e^{-\alpha z} \langle \boldsymbol{a}^+, D\boldsymbol{\phi}^*_z \rangle \, dz \\
&= -\alpha \langle \boldsymbol{a}^+, D\boldsymbol{b}^- \rangle + \alpha \left[e^{-\alpha z} \langle \boldsymbol{a}^+, D\boldsymbol{\phi}^* \rangle\right]_{-\infty}^{\infty} + \alpha^2 \int_{-\infty}^{\infty} e^{-\alpha z} \langle \boldsymbol{a}^+, D\boldsymbol{\phi}^* \rangle \, dz \\
&= -2\alpha \langle \boldsymbol{a}^+, D\boldsymbol{b}^- \rangle + \alpha^2 \int_{-\infty}^{\infty} e^{-\alpha z} \langle \boldsymbol{a}^+, D\boldsymbol{\phi}^* \rangle \, dz.
\end{aligned} \tag{4.70}
$$

Similarly,

$$\int_{-\infty}^{\infty} e^{-\alpha z} \langle\, \boldsymbol{a}^{+}, \boldsymbol{\phi}_z^{*} \,\rangle \, dz = -\langle\, \boldsymbol{a}^{+}, \boldsymbol{b}^{-} \,\rangle + \alpha \int_{-\infty}^{\infty} e^{-\alpha z} \langle\, \boldsymbol{a}^{+}, \boldsymbol{\phi}^{*} \,\rangle \, dz \tag{4.71}$$

holds. Substituting (4.70) and (4.71) into (4.69), we obtain (4.67).

We can prove (4.68) in quite a similar way. ∎

(4.59), (4.60), (4.62) and (4.65) - (4.68) give the proof of this lemma. ∎

Lemmas 4.10 and 4.11 show (2.9) .

5. The proofs of Propositions 4.1 and 4.2.

Throughout this section, we assume $\min \boldsymbol{h}$ is taken sufficiently large and hence $\delta(\boldsymbol{h})$ is small enough.

Let

$$L_j \boldsymbol{v} = D\boldsymbol{v}_{zz} - \theta \boldsymbol{v}_z + F'(P(z-z_j))\boldsymbol{v},$$

$$\widehat{L}\boldsymbol{v} = D\boldsymbol{v}_{zz} - \theta \boldsymbol{v}_z + F'(\boldsymbol{0})\boldsymbol{v}$$

for $\boldsymbol{v} \in \{H^2(\boldsymbol{R}^1)\}^n$, where $z_j = \sum_{k=1}^{j} h_k$ $(j = 1, 2, \cdots, N)$ and $z_0 = 0$ as introduced in the top part of Section 2. Note that each L_j has simple zero eigenvalue together with the eigenfunction $P_z(z-z_j)$. Now, we define a set of cut-off functions $\{\chi_j(z);\ j = 0, 1, \cdots, N\}$ such that $0 \le \chi_j(z) \le 1$, $\sum_{j=0}^{N} \chi_j(z) \equiv 1$ and

$$\chi_j(z) = \begin{cases} 1, & z \in \Omega_j^i = (z_j - \frac{1}{2}h_j + 1, z_j + \frac{1}{2}h_{j+1} - 1), \\ 0, & z \in \Omega_j^o = (-\infty, z_j - \frac{1}{2}h_j - 1) \cup (z_j + \frac{1}{2}h_{j+1} + 1, \infty) \end{cases}$$

for $j = 1, 2, \cdots, N-1$ and

$$\chi_0(z) = \begin{cases} 1, & z \in \Omega_0^i = (-\infty, z_0 + \frac{1}{2}h_1 - 1), \\ 0, & z \in \Omega_0^o = (z_0 + \frac{1}{2}h_{j+1} + 1, \infty), \end{cases}$$

$$\chi_N(z) = \begin{cases} 1, & z \in \Omega_N^i = (z_N - \frac{1}{2}h_N + 1, \infty), \\ 0, & z \in \Omega_N^o = (-\infty, z_N - \frac{1}{2}h_N - 1). \end{cases}$$

LEMMA 5.1. *The spectrum $\Sigma(L(\boldsymbol{h}))$ of $L(\boldsymbol{h})$ consists of sets $\Sigma_1 \cup \Sigma_2$ such that $\Sigma_1 \subset \{\lambda \in \boldsymbol{C};\ |\lambda| \le C\sqrt[4]{\delta(\boldsymbol{h})}\}$ and $\Sigma_2 \subset \{\lambda \in \boldsymbol{C};\ \Re\lambda < -\rho_4\}$ for positive constants C and ρ_4.*

Proof. Define the operator $D(\lambda) = \sum_{j=0}^{N} \chi_j(\lambda - L_j)^{-1}$, and let $\widetilde{\boldsymbol{u}} = D(\lambda)\boldsymbol{f}$, $\boldsymbol{u}_j = (\lambda - L_j)^{-1}\boldsymbol{f}$ for $\lambda \in \rho(L)$ and $\boldsymbol{f} \in X$, where $\rho(L)$ is the resolvent set of L, i.e. $\rho(L) = \boldsymbol{C}\backslash\Sigma(L)$. Note that $\Sigma(L_j) = \Sigma(L)$ for $j = 0, 1, \cdots, N$.

First, we consider $(\lambda - L(\boldsymbol{h}))$ in $L^2(\Omega_j^i)$. For $z \in \Omega_j^i$, $L(\boldsymbol{h})$ is given by

$$L(\boldsymbol{h}) = L_j + B_j,$$

where $B_j = B_j(z;\boldsymbol{h}) = F'(P(z;\boldsymbol{h})) - F'(P(z-z_j))$ is a matrix operator with $|B_j| \le O(e^{-\alpha(z-z_{j-1})} + e^{\beta(z-z_{j+1})}) \le O(e^{-\frac{1}{2}\alpha h_j} + e^{-\frac{1}{2}\beta h_{j+1}}) \le O(\sqrt{\delta})$. Since $\widetilde{\boldsymbol{u}} = \boldsymbol{u}_j$ on Ω_j^i, we have

$$\begin{aligned} (\lambda - L(\boldsymbol{h}))\widetilde{\boldsymbol{u}} &= (\lambda - L(\boldsymbol{h}))\boldsymbol{u}_j \\ &= (\lambda - L_j)\boldsymbol{u}_j + B_j\boldsymbol{u}_j \\ &= \boldsymbol{f} + B_j\boldsymbol{u}_j \\ &= \boldsymbol{f} + B_j\widetilde{\boldsymbol{u}}. \end{aligned} \tag{5.1}$$

Let $\Omega_j^a = (z_j - \frac{3}{4}h_j, z_j - \frac{1}{4}h_j)$ and let $\Omega_j^b = (z_j - \frac{1}{2}h_j - 2, z_j - \frac{1}{2}h_j + 2)$. We shall consider $(\lambda - L(\boldsymbol{h}))$ in $L^2(\Omega_j^b)$. Since $P(z;\boldsymbol{h}) = O(e^{-\alpha(z-z_{j-1})} + e^{\beta(z-z_j)})$, $|P(z;\boldsymbol{h})| \le O(e^{-\frac{1}{2}\alpha h_j} + e^{-\frac{1}{2}\beta h_j}) \le O(\sqrt{\delta})$ holds for $z \in \Omega_j^b$ and $|P(z;\boldsymbol{h})| \le O(\sqrt[4]{\delta})$ holds for $z \in \Omega_j^a$. Hence, $L(\boldsymbol{h})$ is expressed by $L(\boldsymbol{h}) = \widehat{L} + \widehat{B}$, where $\widehat{B} = \widehat{B}(x) = F'(P(z;\boldsymbol{h})) - F'(\boldsymbol{0})$ with the estimate $|\widehat{B}(x)| \le O(\sqrt[4]{\delta})$ on Ω_j^a.

On the ohter hand, L_k are also represented by

$$L_k = \widehat{L} + \widehat{B}_k$$

with $|\widehat{B}_k(x)| \le O(\sqrt[4]{\delta})$ on Ω_j^a and therefore

$$(\lambda - \widehat{L})\boldsymbol{u}_k - \widehat{B}_k \boldsymbol{u}_k = \boldsymbol{f} \tag{5.2}$$

hold for $k = j-1,\ j$, where $\widehat{B}_k = \widehat{B}_k(x) = F'(P(z - z_k)) - F'(\boldsymbol{0})$. Since all the spectra of $\widehat{L}$ is in the left hand side of imaginary axis by the assumption H1), we may assume $(\lambda - \widehat{L})$ is invertible in X for $\lambda \in \boldsymbol{C}$ with $\Re\lambda > -\rho_0$. Let $\widehat{\boldsymbol{u}} = (\lambda - \widehat{L})^{-1}\boldsymbol{f}$. Since $\boldsymbol{u}_k$ $(k = j-1,\ j)$ satisfy (5.2) on Ω_j^a with small $O(\sqrt[4]{\delta})$ purturbations $\widehat{B}_k$ and $\boldsymbol{u}_k$ are bounded on Ω_j^a, we have

$$\|\boldsymbol{u}_k - \widehat{\boldsymbol{u}}\|_{H^1(\Omega_j^b)} \le O(\sqrt[4]{\delta})(\|\widehat{\boldsymbol{u}}\| + \|\boldsymbol{u}_k\| + \|\widetilde{\boldsymbol{u}}\|). \tag{5.3}$$

Let $g_k = \boldsymbol{u}_k - \widehat{\boldsymbol{u}}$. By (5.2), (5.3), it follows on Ω_j^b that

$$\begin{aligned}(\lambda - L(\boldsymbol{h}))\widetilde{\boldsymbol{u}} &= (\lambda - \widehat{L} - \widehat{B})\widetilde{\boldsymbol{u}} \\ &= (\lambda - \widehat{L})(\widehat{\boldsymbol{u}} + \chi_{j-1}g_{j-1} + \chi_j g_j) - \widehat{B}\widetilde{\boldsymbol{u}} \\ &= \boldsymbol{f} + (\lambda - \widehat{L})(\chi_{j-1}g_{j-1} + \chi_j g_j) - \widehat{B}\widetilde{\boldsymbol{u}} \\ &= \boldsymbol{f} + R_{j-1}\boldsymbol{u}_{j-1} + R_j\boldsymbol{u}_j + \widehat{R}\widehat{\boldsymbol{u}} + \widetilde{R}\widetilde{\boldsymbol{u}}\end{aligned} \tag{5.4}$$

for certain bounded operators R_k, $\widetilde{R}$ and $\widehat{R}$ in $L^2(\Omega_j^b)$ with the norms estimated at $O(\sqrt[4]{\delta})$.

(5.1) and (5.4) show that

$$\begin{aligned}(\lambda - L(\boldsymbol{h}))\widetilde{\boldsymbol{u}} &= \boldsymbol{f} + R_{j-1}\boldsymbol{u}_{j-1} + R_j\boldsymbol{u}_j + \widehat{R}\widehat{\boldsymbol{u}} + \widetilde{R}\widetilde{\boldsymbol{u}} \\ &= \boldsymbol{f} + R_{j-1}(\lambda - L_{j-1})^{-1}\boldsymbol{f} + R_j(\lambda - L_j)^{-1}\boldsymbol{f} + \widehat{R}(\lambda - \widetilde{L})^{-1}\boldsymbol{f} + \widetilde{R}D(\lambda)\boldsymbol{f}\end{aligned} \tag{5.5}$$

holds in X. Here, we indetify all operators such as R_j defined on $L^2(\Omega_j^b)$ as bounded operators extended on X with the norms $O(\sqrt[4]{\delta})$. Since $\|(\lambda - L_k)^{-1}\| \le \dfrac{C}{|\lambda|}$ and $\|(\lambda - \widehat{L})^{-1}\| \le \dfrac{C}{|\lambda|}$ hold for $\lambda \in \rho(L)$ and a constant $C > 0$, it follows that

$$\|R_k(\lambda - L_k)^{-1}\|,\ \|\widehat{R}(\lambda - \widehat{L})^{-1}\|,\ \|\widetilde{R}D(\lambda)\| \le \frac{C_5\sqrt[4]{\delta}}{|\lambda|} \tag{5.6}$$

for a constant $C_5 > 0$, which means $Id + R_{j-1}(\lambda - L_{j-1})^{-1} + R_j(\lambda - L_j)^{-1} + \widehat{R}(\lambda - \widetilde{L})^{-1} + \widetilde{R}D(\lambda)$ is invertible and bounded if $|\lambda| > 8C_5\sqrt[4]{\delta}$. This yields that $(\lambda - L(\boldsymbol{h}))$ is invertible and the inverse is given by

$$(\lambda - L(\boldsymbol{h}))^{-1} = D(\lambda)\{Id + R_{j-1}(\lambda - L_{j-1})^{-1} + R_j(\lambda - L_j)^{-1} + \widehat{R}(\lambda - \widetilde{L})^{-1} + \widetilde{R}D(\lambda)\}^{-1} \tag{5.7}$$

for $\lambda \in \rho(L)$ with $|\lambda| > 8C_5\sqrt[4]{\delta}$ and $\Re\lambda > -\rho_4$ for a constant $\rho_4 > 0$.

∎

Let Γ be a closed circle surrounding Σ_1 in the region $\{\lambda \in \boldsymbol{C};\ \Re\lambda > -\rho_4\}$. Then, the projection $Q = Q(\boldsymbol{h})$ from X to $E(\boldsymbol{h})$ is given by

$$Q(\boldsymbol{h}) = \frac{1}{2\pi i}\int_\Gamma (\lambda - L(\boldsymbol{h}))^{-1} d\lambda,$$

where $i = \sqrt{-1}$ and $E(\boldsymbol{h})$ is the eigenspace corresponding to the spectral set Σ_1. Now, we may take $\Gamma = \{\lambda \in \boldsymbol{C};\ |\lambda| = \rho_5\}$ for a constant $\rho_5 > 0$.

Let Q_j be the projection from X to $KerL_j = span\{P_z(z - z_j)\}$, that is,

$$Q_j \boldsymbol{u} = \langle\ \boldsymbol{u}, \boldsymbol{\phi}^*(\cdot - z_j)\ \rangle_{L^2}\, P_z(\cdot - z_j),$$

and define $\widehat{Q} = \sum_{j=0}^{N} Q_j$. Let $\widetilde{Q}$ be the projection operator from X to $span\{P_z(z - z_j);\ j = 0, 1, \cdots, N\}$ such that $\widetilde{Q} = \widehat{Q} + o(1)$ as $\min \boldsymbol{h} \to \infty$. In fact, we can easily construct such a projection $\widetilde{Q}$ satisfying $\widetilde{Q} = \widehat{Q} + O(\sqrt[4]{\delta})$.

Lemma 5.2.

$$\|Q(\boldsymbol{h}) - \widetilde{Q}\| = O(\sqrt[4]{\delta})$$

holds.

Proof. On Γ, $\|R_{j-1}(\lambda - L_{j-1})^{-1} + R_j(\lambda - L_j)^{-1} + \widehat{R}(\lambda - \widetilde{L})^{-1} + \widetilde{R}D(\lambda)\| \leq \dfrac{4C_5\sqrt[4]{\delta}}{|\lambda|} \leq \dfrac{4C_5\sqrt[4]{\delta}}{\rho_5} \leq O(\sqrt[4]{\delta})$ holds by (5.6). Hence, $\{Id + R_{j-1}(\lambda - L_{j-1})^{-1} + R_j(\lambda - L_j)^{-1} + \widehat{R}(\lambda - \widetilde{L})^{-1} + \widetilde{R}D(\lambda)\}^{-1}$ is expanded as

$$\{Id + R_{j-1}(\lambda - L_{j-1})^{-1} + R_j(\lambda - L_j)^{-1} + \widehat{R}(\lambda - \widetilde{L})^{-1} + \widetilde{R}D(\lambda)\}^{-1} = Id + G$$

with $\|G\| \leq O(\sqrt[4]{\delta})$, and from (5.7)

$$(\lambda - L(\boldsymbol{h}))^{-1} = D(\lambda)(Id + G) \tag{5.8}$$

holds for $\lambda \in \Gamma$.

Let $\boldsymbol{f} \in C_0^\infty(\Omega_j^i)$. Since $D(\lambda) = (\lambda - L_j)^{-1}$ on Ω_j^i and $Q_j = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma (\lambda - L_j)^{-1} d\lambda$, we have

$$\begin{aligned}\langle\ Q(\boldsymbol{h})\boldsymbol{u}, \boldsymbol{f}\ \rangle_{L^2} &= \frac{1}{2\pi i}\int_\Gamma \langle\ (\lambda - L_j)^{-1}(Id + G)\boldsymbol{u}, \boldsymbol{f}\ \rangle_{L^2} \\ &= \langle\ Q_j\boldsymbol{u} + G_j\boldsymbol{u}, \boldsymbol{f}\ \rangle_{L^2}\ ,\end{aligned}$$

where $G_j = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma (\lambda - L_j)^{-1} G d\lambda = O(\sqrt[4]{\delta})$. Since $\boldsymbol{f}$ is arbitrary,

$$Q(\boldsymbol{h}) = Q_j + G_j \tag{5.9}$$

holds in $L^2(\Omega_j^i)$.

Next, consider $Q(\boldsymbol{h})$ in $L^2(\Omega_j^b)$. Let $\boldsymbol{f} \in C_0^\infty(\Omega_j^b)$. Then, it follows

$$\begin{aligned}\langle\ Q(\boldsymbol{h})\boldsymbol{u}, \boldsymbol{f}\ \rangle_{L^2} &= \frac{1}{2\pi i}\int_\Gamma \langle\ D(\lambda)(Id + G)\boldsymbol{u}, \boldsymbol{f}\ \rangle_{L^2} \\ &= \frac{1}{2\pi i}\int_\Gamma \sum_{j=0}^{N} \langle\ \chi_j(\lambda - L_j)^{-1}(Id + G)\boldsymbol{u}, \boldsymbol{f}\ \rangle_{L^2}\, d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma \sum_{j=0}^{N} \langle\ (\lambda - L_j)^{-1}(Id + G)\boldsymbol{u}, \chi_j\boldsymbol{f}\ \rangle_{L^2}\, d\lambda \\ &= \sum_{j=0}^{N} \langle\ (Q_j + G_j)\boldsymbol{u}, \chi_j\boldsymbol{f}\ \rangle_{L^2} \\ &= \langle\ \sum_{j=0}^{N} \chi_j(Q_j + G_j)\boldsymbol{u}, \boldsymbol{f}\ \rangle_{L^2}\ .\end{aligned}$$

This shows

$$Q(\boldsymbol{h}) = \sum_{j=0}^{N} \chi_j(Q_j + G_j) \ on \ \Omega_j^b$$

and specially, shows

$$Q(\boldsymbol{h}) = O(\sqrt[4]{\delta}) \tag{5.10}$$

in $L^2(\Omega_j^b)$.

$\widetilde{Q}$ satisfies the same estimates as (5.9) and (5.10), so that the lemma is proved ∎

Lemma 5.2 implies that the spaces $E(\boldsymbol{h}) = Q(\boldsymbol{h})X$ and $\widetilde{Q}X = span\{P_z(z - z_j);\ j = 0, 1, \cdots, N\}$ are homeomorphic each other and hence $\dim E(\boldsymbol{h}) = N + 1$ holds.

Let a projection $R(\boldsymbol{h})$ be $Id - Q(\boldsymbol{h})$ and $E^{\perp}(\boldsymbol{h}) = R(\boldsymbol{h})X$.

Lemma 5.3.

$$\|L(\boldsymbol{h})^{-1}|_{E^{\perp}(\boldsymbol{h})}\| \le C$$

holds uniformly for $\boldsymbol{h}$ *with sufficiently large* $\min \boldsymbol{h}$.

Proof. Let $f(\lambda) = \langle\ (\lambda - L(\boldsymbol{h}))^{-1}\boldsymbol{u}, \boldsymbol{v}\ \rangle_{L^2}$ for $\boldsymbol{u} \in E^{\perp}(\boldsymbol{h})$ and $\boldsymbol{v} \in X$. Then, $f(\lambda)$ is holomorphic funciton of $\lambda \in D[\Gamma] = \{\lambda \in \boldsymbol{C};\ |\lambda| < \rho_5\}$ because $L(\boldsymbol{h})$ is invertible on the space $E^{\perp}(\boldsymbol{h})$. Since $f(\lambda)$ has an estimate

$$|f(\lambda)| \le \frac{C_7}{|\lambda|}\|\boldsymbol{u}\| \cdot \|\boldsymbol{v}\|$$

by (5.7) for a constant $C_7 > 0$, $|f(\lambda)| \le \dfrac{C_7}{\rho_5}\|\boldsymbol{u}\| \cdot \|\boldsymbol{v}\|$ holds on Γ. Therefore, the maximum principle for the holomorphic functions yields

$$|f(\lambda)| \le \frac{C_7}{\rho_5}\|\boldsymbol{u}\| \cdot \|\boldsymbol{v}\|$$

for any $\lambda \in D[\Gamma]$ and specially for $\lambda = 0$, which give the proof. ∎

Since $L(\boldsymbol{h})P_z(z - z_j) = O(\delta)$ hold, we have $L(\boldsymbol{h})R(\boldsymbol{h})P_z(\cdot - z_j) = O(\delta)$. Lemma 5.3 yields

$$R(\boldsymbol{h})P_z(\cdot - z_j) = O(\delta). \tag{5.11}$$

Let $\boldsymbol{\psi}_j = Q(\boldsymbol{h})P_z(\cdot - z_j)$. Then, $\boldsymbol{\psi}_j \in E(\boldsymbol{h})$ and we see

$$\boldsymbol{\psi}_j = P_z(\cdot - z_j) - R(\boldsymbol{h})P_z(\cdot - z_j) = P_z(\cdot - z_j) + O(\delta)$$

by (5.11). Since $\boldsymbol{\psi}_0, \cdots, \boldsymbol{\psi}_N$ are obviously linearly independent and $\dim E(\boldsymbol{h}) = N + 1$, these give the basis of $E(\boldsymbol{h})$, that is,

$$E(\boldsymbol{h}) = span\{\boldsymbol{\psi}_0, \cdots, \boldsymbol{\psi}_N\}.$$

Thus, we know $\Sigma_1 \subset \{\lambda \in \boldsymbol{C};\ |\lambda| \le C\delta\}$ for a constant $C > 0$ because $Q(\boldsymbol{h})\boldsymbol{\psi}_j = O(\delta)$.

For the adjoint operator $L^*(\boldsymbol{h})$, quite a similar properties to $L(\boldsymbol{h})$ hold. That is, there exist $\{\boldsymbol{\phi}_0^*, \cdots, \boldsymbol{\phi}_N^*\}$ such that $\boldsymbol{\phi}^* = \boldsymbol{\phi}^*(\cdot - z_j) + O(\delta)$ and $E^*(\boldsymbol{h}) = span\{\boldsymbol{\phi}_0^*, \cdots, \boldsymbol{\phi}_N^*\}$. Since

$$\langle\ \boldsymbol{\psi}_j, \phi_k^*\ \rangle_{L^2} = \begin{cases} 1 + O(\delta) & (j = k), \\ O(\delta) & (j \ne k), \end{cases}$$

we can easily construct $\{\boldsymbol{\phi}_0, \cdots, \boldsymbol{\phi}_N\}$ in Propotitions 4.1 and 4.2 by slightly modifying $\{\boldsymbol{\psi}_0, \cdots, \boldsymbol{\psi}_N\}$.

6. Discussions and extensions.

In this section, we will state several extensions of the results of this paper.

First, we would like to mention that the method of proofs developed in this paper can be easily extended to higher dimensional problems. In fact, we will show the repulsiveness of the spike solutions for the Gierer-Meinhardt models on two dimensional space in the forthcoming paper [14].

On the other hand, the idea based on pulse interactions contributes powerfully to analyze various transient behaviors. For example, we will see in the forthcoming papars [10], [11] that the approach of weakly interacting pulses stated in this paper is very useful to analyze the self-replicating behaviors appearing in the Gray-Scott model and/or the reflection of traveling pulses in some reaction-diffuision systems.

Finally, we emphasize that our problem (1.3) in this paper is easily extended to the problem with small perturbations like

$$\boldsymbol{u}_t = D\boldsymbol{u}_{xx} + F(\boldsymbol{u}) + \epsilon g(\boldsymbol{u}, \boldsymbol{u}_x) \tag{6.1}$$

for small ϵ. The problem of this type includes many important problems such as the bifurcation problems of homoclinic and/or heteroclinic orbits. Let us only consider the problems related to heteroclinic orbits here. Suppose the equation (6.1) with $\epsilon = 0$ has a stable 1-front solution, say $P(x)$, which satisfies $\theta = 0$ and $P(x) \to e^{-\alpha|x|}\boldsymbol{a}^{\pm} + P^{\pm}$ as in Section 2.2. $P(x)$ is a heteroclinic solution for the unperturbed equation. Then, the solution $\boldsymbol{u}(t, x)$ of (6.1) remains close to $P(x - l)$ and

$$\begin{aligned}\dot{l} &= -\epsilon \langle\, g(P(x-l), P_x(x-l)), \boldsymbol{\phi}^*(x-l)\,\rangle_{L^2} + O(\epsilon^2)\\ &= -\epsilon \langle\, g(P(x), P_x(x)), \boldsymbol{\phi}^*(x)\,\rangle_{L^2} + O(\epsilon^2)\\ &= -\epsilon C + O(\epsilon^2)\end{aligned}$$

holds for the constant $C = \langle\, g(P(x), P_x(x)), \boldsymbol{\phi}^*(x)\,\rangle_{L^2}$ since g includes no space variable x. This is easily proved in quite a similar way to this paper. Thus, we can know what kind of traveling front bifurcates by the perturbation ϵg.

The bifurcation of homoclinic solutions is also dealt with similarly. Let us consider the solution (6.1) with the initial function $\boldsymbol{u}_0(x)$ close to $P(x - l_0) + P(-x + l_0 + h_0) - P^-$ for sufficiently large h_0. Then we can show that the solution $\boldsymbol{u}(t, x)$ remains close to $P(x - l(t)) + P(-x + l(t) + h(t))$ and

$$\dot{l} = -M^+ e^{-\alpha h} - \epsilon C + O(\delta^2 + \epsilon^2), \tag{6.2}$$

$$\dot{h} = 2M^+ e^{-\alpha h} + 2\epsilon C + O(\delta^2 + \epsilon^2) \tag{6.3}$$

hold, where M^+ is the constant stated in Theorem 2.5. Then, we can know directly from (6.3) that when $M^+ > 0 (< 0)$ and $C < 0 (> 0)$, (6.3) has one (un)stable equilibrium, which means the existence of (un)stable pulse solution corresponding to the homoclinic orbit with respect to P^-. Thus, we find the stability of bifurcating traveling pulse solution crucially depends on the repulsiveness ($M^+ > 0$) or attractivity ($M^+ < 0$) of pulses.

We have found that the approach of weakly interacting pulses stated in this paper is very effective also to the study of bifurcation problems for homoclinic orbits in ODEs. In this field, there have been a lot of important works from the dynamical system point of view ([23] and see the references). For this research, the relation between the theories for dynamical systems in ODE and the method in this paper is left as an important problem to be clarified.

REFERENCES

[1] N. Alikakos, P.W. Bates and G. Fusco, Slow motion for the Cahn-Hilliard equation in one space dimension, J. Differential equations, 90 (1991), 81-135.

[2] J. Carr and R. L. Pego, Metastable patterns in solutions of $u_t = \varepsilon^2 u_{xx} + f(u)$, Comm. Pure Appl. Math., 42 (1989), 523-576.

[3] A. Doelman, R. A. Gardner and T. J. Kaper, Stability analysis of singular patterns in the 1-D Gray-Scott model, physica D, 122 (1998), 1-36.

[4] S.-I. Ei, Interaction of pulses in FitzHugh-Nagumo equations, Reaction-Diffusion Equations and Their Applications and Computational Aspects, China-Japan Symposium (eds. T-T. Li, M. Mimura, Y. Nishiura, Q-X. Ye) pp.6-13, World Scientific, 1994.

[5] J.W. Evans, Nerve axon equations: I Linear approximations, Indiana Univ. Math. J., 21 (1972), 877-955.
[6] J.W. Evans, Nerve axon equations: II Stability at rest, Indiana Univ. Math. J., 22 (1972), 75-90.
[7] J.W. Evans, Nerve axon equations: III Stability of the nerve impulse, Indiana Univ. Math. J., 22 (1972), 577-593.
[8] J.W. Evans, Nerve axon equations: IV The stable and unstable impulses, Indiana Univ. Math. J., 24 (1975), 1169-1190.
[9] S.-I. Ei, H. Ikeda and E. Yanagida, in preparation.
[10] S.-I. Ei, M. Mimura and M. Nagayama, Reflection of travelling pulses in reaction-diffusion systems, in preparation.
[11] S.-I. Ei, Y. Nishiura and B. Sandstede, Pulse-interaction approach to self-replicating dynamics in reaction-diffusion systems, in preparation.
[12] S.-I. Ei, Y. Nishiura and K. Ueda, 2^n- splitting or edge-splitting ?, preprint.
[13] S.-I. Ei and T. Ohta, The montion of interacting pulses, Physical Review E, 50 No. 6 (1994), 4672-4678.
[14] S.-I. Ei and J. Wei, Dynamics of Metastable Localized Patterns and its Application to the Interaction of Spike Solutions for The Gierer-Meinhardt System in 2 dimensinal space, preprint.
[15] G. Fusco and J. Hale, Slow motion manifold, dormant instability and singular perturbations, J. Dynamics and Differential Equations, 1 (1989), 75-94.
[16] , K.A. Gorshkov and L.A. Ostrovsky, Physica 3D (1981)
[17] C. K. R. T. Jones, Stability of the traveling wave solution of the FitzHugh-Nagumo system, Trans. A. M. S. 286(1984), 431-469.
[18] P. C. Fife and J. B. Mcleod, The approach of solutions of nonlinear diffusion equations to travelling front solutions, Arch. Rat. Mech. Anal., 65 (1977), 335-361.
[19] D. Henry, Geometric theory of semilinear parabolic equations, Lecture Notes in Mathematics 840, Springer-Verlag, 1981.
[20] Y. Kan-on, Parameter dependence of propagation speed of travelling waves for competition-diffusion equations, SIAM J. Appl. Math. 26 (1995), 340-363.
[21] K. Kawasaki and T. Ohta, Kink dynamics in one-dimensional nonlinear systems, Physica 116A (1982), 573-593.
[22] L.D. Landau and E.M. Lifshitz, Quantum Mechanics, Pergamon, Oxford, 1977.
[23] S. Nii, A topoligical proof of stability of multiple- front solutions of the FitzHugh-Nagumo equations, J. Dynam. Diff.Eqs. 11 (1999), 515-555.
[24] Y. Nishiura and D. Ueyama, A skelton structure of self-replicating dynamics, Physica D 130 (1999), 73-104.
[25] J.E. Pearson, Complex patterns in a simple system, Science vol. 216 (1993), 189-192.
[26] B. Sandstede, Stability of multiple-pulse solutions, Trans. Amer. Math. Soc., 350 (1998), 429-472.
[27] B. Sandstede, Weak interaction of pulses, in preparation.
[28] M. Schatzmann, in preparation.
[29] I. Tazaki and G. Matsumoto, Mechanism of nerve impulses, Sangyo Tosyo, 1975 (in Japanese).
[30] E. Yanagida, Stability of fast travelling pulse solutions of the FitzHugh-Nagumo equations, J. Math. Biol., 22(1985), 81-104.
[31] H. Yamada and K. Nozaki, Interaction of pulses in dissipative systems, Prog. Theo. Physics 84(1990), 801-809.